Lie groups over non-discrete topological fields

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Abstract

We generalize the classical construction principles of infinite-dimensional real (or complex) Lie groups to the case of Lie groups over non-discrete topological fields. In particular, we discuss linear Lie groups, mapping groups, test function groups, diffeomorphism groups, and weak direct products of Lie groups. The specific tools of differential calculus required for the Lie group constructions are developed. Notably, we establish differentiability properties of composition and evaluation, as well as exponential laws for function spaces. We also present techniques to deal with the subtle differentiability and continuity properties of non-linear mappings between spaces of test functions. Most of the results are independent of any specific properties of the topological vector spaces involved; in particular, we can deal with real and complex Lie groups modeled on non-locally convex spaces.

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Key words: Infinite-dimensional Lie groups, continuous inverse algebras, linear Lie groups, mapping groups, test function groups, diffeomorphism groups, weak direct products, non-locally convex spaces, direct sums, patched topological vector spaces, almost local mappings, direct limits, ultrametric calculus, convenient differential calculus, topological fields, local fields, non-archimedean analysis, exponential law, composition map, evaluation map, Boman's theorem

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Introduction

Most of the known examples of infinite-dimensional real or complex Lie groups can be subsumed under (at least) one of the following main classes of Lie groups: 1. linear Lie groups; 2. mapping groups; 3. diffeomorphism groups. In the present article, we show that the general construction principles underlying these classes of Lie groups work just as well beyond the real and complex cases. Thus, we are able to discuss linear Lie groups and groups of continuous Lie group-valued mappings over arbitrary non-discrete topological fields; groups of smooth Lie group-valued mappings on finite-dimensional smooth manifolds over locally compact fields; and diffeomorphism groups of paracompact finite-dimensional smooth manifolds over local fields (of arbitrary characteristic). In the real and complex cases, it becomes possible to construct Lie groups modeled on arbitrary (not necessarily locally convex) topological vector spaces.

A fourth main class of infinite-dimensional Lie groups are Lie groups obtained from direct limit constructions, in particular direct limits of finite-dimensional Lie groups (see [63], [64], [21], [29]). Direct limits of finite-dimensional Lie groups over local fields have been discussed in [29] (cf. also [21]). Here, we construct weak direct products of (finite- or infinite-dimensional) Lie groups over valued fields, generalizing the discussion of weak direct products of Lie groups modeled on real or complex locally convex spaces from [22].

Our studies are based on a concept of C^k-maps (and smooth maps) between open subsets of topological vector spaces over non-discrete topological fields introduced in [3], where more generally an axiomatic approach to differential calculus over arbitrary infinite fields (and suitable commutative rings) is developed. A map between open subsets of *locally convex* real topological vector spaces is of class C^k in the sense considered here if and only if it is a C^k-map in the sense of Michal-Bastiani, *i.e.*, a C^k_c-map in the terminology of Keller's monograph [44] (see [3]). C^k-maps in the latter sense have been used as the basis of differential calculus and infinite-dimensional Lie theory by many authors (see, *e.g.*, [10], [18]–[23], [25], [27], [29]–[34], [37], [43], [57]–[59], [65]–[67], [70], and [80]). Furthermore, a map between open subsets of complex locally convex spaces is complex C^∞ if and only if it is complex analytic in the usual sense (as in [8]).

Taken together, the papers [3], [28] and the present work develop, from first principles, a comprehensive theory of differential calculus and Lie groups over arbitrary non-discrete topological fields. In [3] already mentioned, an exposition of differential calculus over topological fields and the corresponding basic theory of manifolds and Lie groups is given; this article is directed to a broad audience including readers without prior knowledge of differential calculus over topological fields. In [28] (needed here only for the discussion of diffeomorphism groups), implicit (and inverse) function theorems for C^k-maps over complete valued fields are established. The present article, then, provides concrete examples of Lie groups over topological fields, and develops the specific aspects of differential calculus required for this purpose. Further papers related to the "General Differential Calculus" over topological fields are [2], [4], [5], [24], [26], [29] and [30]. In [26], it was shown that

every finite-dimensional smooth p-adic Lie group is a p-adic analytic Lie group in the usual sense. Important aspects of differential geometry over topological fields have been worked out in [2]. Jordan theoretic applications are described in [4] and [5].

The article commences with a brief introduction to differential calculus over non-discrete topological fields (Section 1), and ends with appendices covering material which is best taken on faith on a first reading, and whose presentation within the text would have distracted from the main line of thought. Apart from these sections, the main body of the text is divided into three parts, devoted to the three main classes and construction principles of infinite-dimensional Lie groups described above:

I. Linear Lie groups

Paradigms of real or complex Lie groups are *linear* Lie groups, *i.e.*, unit groups of unital Banach algebras (or other well-behaved topological algebras) and their Lie subgroups (see [20], [38], [41, Ch. 5], [56]). We begin our studies with a discussion of linear Lie groups over topological fields (Section 2), as this only requires a minimum of technical machinery. If $\mathbb{K}$ is a non-discrete topological field, a good class of topological $\mathbb{K}$-algebras to look at are the *continuous inverse algebras* (or CIAs), *viz.* unital associative topological $\mathbb{K}$-algebras A such that the group of units $A^\times$ is open in A and the inversion map $\iota\colon A^\times \to A$, $a \mapsto a^{-1}$ is continuous. We describe examples of CIAs and construction principles for CIAs over arbitrary non-discrete topological fields. Since the unit group $A^\times$ is a $\mathbb{K}$-Lie group for any continuous inverse algebra A (Proposition 2.2), we thus always have a certain supply of $\mathbb{K}$-Lie groups, for any $\mathbb{K}$ (beyond the trivial examples, the additive groups of topological $\mathbb{K}$-vector spaces). Algebras of continuous or differentiable maps on compact topological spaces or compact manifolds, with values in a CIA, are again CIAs (Proposition 5.7). For further typical examples of CIAs in the real or complex cases, see [20], [23], [36, 1.15], [81].

II. Mapping groups and related constructions

The second widely studied class of infinite-dimensional real (or complex) Lie groups are the mapping groups. Typical examples are the "loop groups" $C(\mathbb{S}^1, G)$ and $C^\infty(\mathbb{S}^1, G)$, where G is a finite-dimensional real (or complex) Lie group [70]. More generally, let G be a real or complex Lie group modeled on a locally convex space, $r \in \mathbb{N}_0 \cup \{\infty\}$, and M be a finite-dimensional smooth manifold (or topological space if $r = 0$). Among the types of mapping groups encountered in the literature, we mention: the groups $C^r(M, G)$ of G-valued C^r-maps, for compact M; the groups $C^r_K(M, G)$ of G-valued C^r-maps supported in a compact set $K \subseteq M$; and, for σ-compact manifolds M, the "test function groups" $C^r_c(M, G) := \bigcup_K C^r_K(M, G)$ of compactly supported G-valued C^r-maps (see [1], [19], [27], [47], [59], [64], [66], [67]).

In the second main part of this article, we construct Lie group structures on analogous mapping groups in the case of Lie groups over topological fields. The results include:

Groups of continuous mappings. *If $\mathbb{K}$ is a non-discrete topological field, G a $\mathbb{K}$-Lie group, X a topological space, and $K \subseteq X$ a compact subset, then the group*

$$C_K(X,G) := \{\gamma \in C(X,G)\colon \gamma|_{X\setminus K} = 1\}$$

of continuous G-valued mappings supported in K can be made a $\mathbb{K}$-Lie group modeled on $C_K(X, L(G))$, in a natural way. In particular, $C(K,G) = C_K(K,G)$ is a $\mathbb{K}$-Lie group, for every compact topological space K and any $\mathbb{K}$-Lie group G.

Groups of differentiable mappings. *Let $\mathbb{F}$ be a locally compact, non-discrete topological field, $\mathbb{K}$ be a topological extension field of $\mathbb{F}$, G be a $\mathbb{K}$-Lie group, $r \in \mathbb{N}_0 \cup \{\infty\}$, M be a finite-dimensional $C^r_{\mathbb{F}}$-manifold, $K \subseteq M$ a compact subset, and*

$$C^r_K(M,G) := \{\gamma \in C^r(M,G)\colon \gamma|_{M\setminus K} = 1\}$$

be the group of G-valued $C^r_{\mathbb{F}}$-functions on M supported in K. Then $C^r_K(M,G)$ is a $\mathbb{K}$-Lie group modeled on $C^r_K(M, L(G))$, in a natural way. If M is paracompact and the topology on $\mathbb{K}$ arises from an absolute value, then also the group $C^r_c(M,G) := \bigcup_K C^r_K(M,G)$ of G-valued test functions of class $C^r_{\mathbb{F}}$ is a $\mathbb{K}$-Lie group, modeled on $C^r_c(M, L(G))$.

(See Sections 5 and 9). Typically, we might choose $\mathbb{K} := \mathbb{F}$ here, or $\mathbb{F} := \mathbb{R}$, $\mathbb{K} := \mathbb{C}$. As the basis for the construction of Lie group structures on mapping groups, we study continuity and differentiability properties of mappings between function spaces. The three cases of interest (mappings between spaces of C_K-functions, C^r_K-functions, and C^r_c-functions, respectively) are discussed in turn in Sections 3, 4, resp., 8 and 10. For simplicity, let us assume that $\mathbb{K} := \mathbb{F}$ now, for the remainder of the introduction. The results obtained subsume, for example, that the mappings

$$\begin{aligned} C^\infty_K(M,g)\colon &\quad C^\infty_K(M,U) \to C^\infty_K(M,F), \quad \gamma \mapsto g\circ\gamma \qquad\qquad \text{and}\\ f_*\colon &\quad C^\infty_K(M,U) \to C^\infty_K(M,F), \quad \gamma \mapsto f\circ(\mathrm{id}_M,\gamma) \end{aligned}$$

are of class $C^\infty_{\mathbb{K}}$, for any $C^\infty_{\mathbb{K}}$-maps $g : U \to F$ and $f : M \times U \to F$ such that $g(0) = 0$ and $f|_{(M\setminus K)\times\{0\}} = 0$, where $\mathbb{K}$ is a locally compact, non-discrete topological field, E and F are topological $\mathbb{K}$-vector spaces, $U \subseteq E$ an open zero-neighbourhood, M a finite-dimensional $C^\infty_{\mathbb{K}}$-manifold, and $K \subseteq M$ a compact subset. For paracompact M, analogous conclusions are valid for $C^\infty_c(M,g)$ and $f_* : C^\infty_c(M,U) \to C^\infty_c(M,F)$. More generally, results of the preceding type are established for mappings between spaces of sections in vector bundles, whose fibres are arbitrary topological vector spaces (Appendix F). For the real locally convex case, the reader may compare Michor [57], in particular his "Ω-Lemma" [57, Thm. 8.7] for finite-dimensional real vector bundles; [19] (for maps between spaces of test functions), and [32]. It is a peculiarity of differential calculus over general topological fields that, when we are trying to prove differentiability properties of f_* (or related results), *parameter-dependent* variants invariably pop up in the natural inductive arguments, even when we are only interested in the case of f_* (not involving parameters). On the one hand, this makes the proofs more complicated; but, on the other hand, we are rewarded

with stronger, parameter-dependent versions of the basic results (like our "Ω-Lemma with Parameters", Theorem F.23), which are novel even in the real locally convex case.

Topologies on spaces of test functions. If $\mathbb{K} \neq \mathbb{C}$ is locally compact, E a topological $\mathbb{K}$-vector space and M a σ-compact, finite-dimensional $C^r_{\mathbb{K}}$-manifold, we equip $C^r_c(M,E) = \varinjlim C^r_K(M,E)$ with the topology making it the direct limit of its subspaces $C^r_K(M,E)$ in the category of topological $\mathbb{K}$-vector spaces. Although little is known on direct limits of general topological vector spaces (in contrast to the real or complex locally convex case, which has been studied extensively), we get a perfectly firm grip on the topology of $C^r_c(M,E)$ by showing that the linear map

$$\rho\colon C^r_c(M,E) \to \bigoplus_{i\in I} C^r(U_i,E), \quad \rho(\gamma) := (\gamma|_{U_i})_{i\in I} \tag{1}$$

is a topological embedding onto a closed vector subspace, for any locally finite cover $(U_i)_{i\in I}$ of M by relatively compact, open subsets U_i. Here, the direct sum is equipped with the box topology, which is extremely simple to work with.

If $\mathbb{K} = \mathbb{C}$, or if M is merely paracompact, then the topology making $C^r_c(M,E)$ the direct limit topological vector space $\varinjlim C^r_K(M,E)$ is too elusive to be useful for us. Instead of excluding these cases altogether from our considerations, we simply replace the direct limit topology on $C^r_c(M,E)$ with the topology induced by ρ, which turns out to be independent of the choice of open cover $(U_i)_{i\in I}$. This enables us to carry out most of our constructions also in the complex case, and also for paracompact manifolds (see Proposition 8.13 and Remark 8.16 for further discussions of the box topology and explanations why we prefer to use it). Note that, for a non-locally convex complex topological vector space E, the space $C^r_c(M,E)$ of compactly supported E-valued $C^r_{\mathbb{C}}$-maps need not be reduced to the locally constant functions. For example, there are non-zero compactly supported $C^\infty_{\mathbb{C}}$-maps $\mathbb{C} \to E$, for suitable E (see [24]). As we do not have cut-off functions (nor partitions of unity) available in the complex case, we have to proceed with particular care. In Section 10, their use cannot be avoided any longer, and the complex case has to be excluded then.

Mappings between direct sums. The embedding ρ from (1) allows us to reduce the study of continuity and differentiability properties of mappings between spaces of test functions (or compactly supported sections in vector bundles) almost entirely to the study of differentiability properties of mappings of the form

$$\oplus_{i\in I} f_i : \bigoplus_{i\in I} U_i \to \bigoplus_{i\in I} F_i, \quad (x_i)_{i\in I} \mapsto (f_i(x_i))_{i\in I}$$

on open boxes $\bigoplus_{i\in I} U_i$ in direct sums $\bigoplus_{i\in I} E_i$ of topological $\mathbb{K}$-vector spaces. In Section 6, for arbitrary valued fields $\mathbb{K}$, we show that a mapping $\oplus_{i\in I} f_i$ as before is $C^k_{\mathbb{K}}$ provided each f_i is $C^k_{\mathbb{K}}$. Although the proof of the special case where I is countable and we are dealing with real or complex locally convex topological vector spaces is almost trivial (see [22]), the proof of the general case requires a substantial amount of work. In order to control

simultaneously, for each $i \in I$, the dependence on the parameter t of the extended difference quotient maps

$$f_i^{[1]} \colon U_i \times E_i \times \mathbb{K} \supseteq U_i^{[1]} \to F_i\,, \quad (x_i, y_i, t) \mapsto f_i^{[1]}(x_i, y_i, t)$$

and, more generally, analogous parameters in the mappings $f_i^{[j]}$, where $j \in \mathbb{N}$, $j \leq k$, which are encountered in the inductive arguments, we are forced to investigate in some depth the symmetry properties of the higher difference quotient maps

$$f_i^{[j]} \colon U_i^{[j]} \to F_i\,,$$

which are rather complicated functions depending on $2^{j+1}-1$ variables (Proposition 6.9). In the case of locally compact fields, the dependence on parameters can be controlled more easily by means of compactness arguments, which in fact permit us to formulate stronger results, involving additional parameters (Proposition 6.10). As a first straightforward application, mappings between direct sums are used to define a Lie group structure on (countable or uncountable) weak direct products $\prod^*_{i\in I} G_i$ of Lie groups G_i over a valued field $\mathbb{K}$, based on the box topology on direct sums (Section 7). As we shall see later, such groups are encountered quite frequently in the case of ultrametric fields; for example, they shall play an important role in our discussion of diffeomorphism groups over local fields. Weak direct products of Lie groups modeled on real or complex locally convex spaces, based on locally convex direct sums, have first been considered in [22].

In the real finite-dimensional case, embeddings in locally convex direct sums are implicit in [57, §4.7] in connection with descriptions of the "$\mathcal{D}$-topology" on mapping spaces, which are used there for similar purposes. The usefulness of embeddings into real and complex locally convex direct sums for the study of mappings between spaces of test functions (and compactly supported sections) has been pointed out explicitly in [32], [33]. The arguments in [57] (based on jet bundles) are restricted to vector bundles over finite-dimensional bases. We approach function spaces and mappings between them in a more direct way. This allows us, for instance, to prove smoothness of the pushforward

$$f_* \colon C^\infty(M, E) \to C^\infty(M, F)$$

in the case of a globally defined $C^\infty_{\mathbb{K}}$-map $f\colon M \times E \to F$ in utmost generality, namely, for $\mathbb{K}$ an arbitrary topological field, M a $C^\infty_{\mathbb{K}}$-manifold modeled on an arbitrary topological $\mathbb{K}$-vector space, and E, F arbitrary topological $\mathbb{K}$-vector spaces (Proposition 4.16).

Differentiability properties of almost local mappings. In order to motivate the most general results we have to offer which exploit embeddings into direct sums (presented in Section 10), we recall from [31] that the self-map

$$f \colon C^\infty_c(\mathbb{R}, \mathbb{R}) \to C^\infty_c(\mathbb{R}, \mathbb{R}), \quad f(\gamma) := \gamma \circ \gamma \,-\, \gamma(0)$$

of the space $C^\infty_c(\mathbb{R}, \mathbb{R})$ of real-valued test functions on the real line is discontinuous (and hence not smooth), although the restriction of f to $C^\infty_K(\mathbb{R}, \mathbb{R})$ is smooth, for every compact

subset $K \subseteq \mathbb{R}$. In the real and complex case, and, more generally, in the case of locally compact topological fields $\mathbb{K}$, this poses the question whether it is possible to specify simple and easily verified *additional properties* ensuring that a mapping

$$f\colon C^r_c(M,E) \to C^s_c(N,F)$$

between spaces of vector-valued test functions is indeed $C^k_{\mathbb{K}}$, provided the restriction of f to $C^r_K(M,E)$ is $C^k_{\mathbb{K}}$ for each compact subset $K \subseteq M$ (and likewise for mappings between spaces of compactly supported sections, or open subsets thereof).

Generalizing the real locally convex case (see [27], [32], [33]), we show that also in the case of general locally compact fields $\mathbb{K} \neq \mathbb{C}$, the requirement that f be *almost local* is a suitable additional property on f (Theorem 10.4), meaning that there exist locally finite, relatively compact open covers $(U_i)_{i\in I}$ of M and $(V_i)_{i\in I}$ of N such that $f(\gamma)|_{V_i}$ only depends on $\gamma|_{U_i}$, for any i. The class of almost local maps includes most mappings of interest. For example, in the case $M = N$, every pushforward of sections associated with a fibre-preserving bundle map is almost local. Furthermore, all mappings encountered in the construction of Lie group structures on diffeomorphism groups of σ-compact finite-dimensional real C^∞-manifolds are (locally) almost local (see [25] and [33], where diffeomorphism groups are discussed along lines independent of the earlier work [57]).

For a highly developed theory of spaces (and manifolds) of mappings and mappings between these in the "convenient setting of analysis" (based on Mackey complete real or complex locally convex spaces), which is inequivalent to the setting of analysis we are working in here, see [17], [47] and further works by the same authors.

III. Diffeomorphism groups

The third main part of the article is devoted to diffeomorphism groups of finite-dimensional manifolds over local fields, and related material. We begin with a discussion of continuity and differentiability properties of evaluation and composition of maps in the context of locally compact fields $\mathbb{K}$ (Section 11). Among variants and related results, we show in particular that the evaluation map

$$\varepsilon\colon C^r(M,E)\times M \to E, \quad \varepsilon(\gamma,x) := \gamma(x)$$

is of class $C^r_{\mathbb{K}}$, for every finite-dimensional $C^r_{\mathbb{K}}$-manifold M and topological $\mathbb{K}$-vector space E (Proposition 11.1), and that the composition map

$$\Gamma\colon C^{r+k}(U,E)\times C^r_K(M,U) \to C^r(M,E), \quad \Gamma(\gamma,\eta) := \gamma\circ\eta$$

is of class $C^k_{\mathbb{K}}$, for any $r,k \in \mathbb{N}_0\cup\{\infty\}$, finite-dimensional $C^r_{\mathbb{K}}$-manifold M, topological $\mathbb{K}$-vector space E, compact subset $K \subseteq M$, and open subset U of a finite-dimensional $\mathbb{K}$-vector space F (Proposition 11.2). We then turn to the exponential law for smooth mappings (Section 12). Given any topological field $\mathbb{K}$, topological $\mathbb{K}$-vector space E, $r,k \in \mathbb{N}_0\cup\{\infty\}$ and arbitrary $C^{r+k}_{\mathbb{K}}$-manifolds M and N, we show that

$$f^\vee\colon M \to C^r(N,E), \quad f^\vee(x)(y) := f(x,y)$$

is of class $C^k_{\mathbb{K}}$ for all $f \in C^{r+k}(M \times N, E)$, and that the mapping

$$\Phi\colon C^{r+k}(M \times N, E) \to C^k(M, C^r(N,E)), \quad \Phi(f) := f^\vee \tag{2}$$

is a continuous linear injection (Lemma 12.1). If $\mathbb{K}$ is locally compact and N is finite-dimensional (but M arbitrary), we show that Φ is an isomorphism of topological vector spaces, in the case $r = k = \infty$ (Proposition 12.2).[1] Similar (slightly weaker) conclusions hold if both M and N are modeled on metrizable topological vector spaces and $\mathbb{K}$ is $\mathbb{R}$ or an ultrametric field (Proposition 12.6). To deduce the surjectivity of Φ in the metrizable case, we make use of techniques of convenient differential calculus (already mentioned), suitably adapted to non-locally convex or ultrametric analysis by means of preparatory results provided in [3]. We can only broach on the subject of "ultrametric convenient differential calculus" here, and have to confine ourselves to what is actually needed for the concrete purpose. A further application of the exponential laws is given in Appendix E. Combining the latter and an ultrametric analogue of Grothendieck's Theorem (relating smoothness and weak smoothness of suitable maps) provided in Appendix D, it is shown there that Boman's theorem will hold for mappings $f : E \supseteq U \to F$ from an open subset of a metrizable topological vector space E over a local field $\mathbb{K}$ to a Mackey complete locally convex topological $\mathbb{K}$-vector space F provided Boman's theorem holds for all mappings $f\colon \mathbb{K}^2 \to \mathbb{K}$. Recall that Boman's classical theorem [9, Thm. 1] asserts that a map $f : \mathbb{R}^n \to \mathbb{R}$ is smooth if and only if $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is smooth for each smooth curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$. Whether Boman's theorem transfers to maps $f\colon \mathbb{K}^2 \to \mathbb{K}$ is still unknown.

In the final Sections 13 and 14, which can be read independently, we describe two approaches to diffeomorphism groups of finite-dimensional smooth manifolds over local fields. The first approach (Section 13) produces a Lie group structure on $\mathrm{Diff}^\infty(M)$, for every paracompact, finite-dimensional smooth manifold M over a local field $\mathbb{K}$. The second approach (Section 14) is restricted to σ-compact M. It produces *two* Lie group structures on $\mathrm{Diff}^\infty(M)$ (one of which coincides with the one constructed in Section 13). Both approaches make use of many of the results and techniques prepared before (and hence also illustrate the usefulness and typical applications of these results).

First approach (Section 13). Let M be a finite-dimensional, paracompact $C^\infty_{\mathbb{K}}$-manifold over a local field $\mathbb{K}$, and $\mathrm{Diff}^\infty(M)$ be the group of all $C^\infty_{\mathbb{K}}$-diffeomorphisms of M. Our first construction of a Lie group structure on $\mathrm{Diff}^\infty(M)$ relies on the fact M is a disjoint union of a family $(B_i)_{i\in I}$ of open and compact balls $B_i \subseteq M$ (i.e., subsets $B_i \subseteq M$ which are $C^\infty_{\mathbb{K}}$-diffeomorphic to balls in $\mathbb{K}^d$ with respect to the supremum-norm). Motivated by this decomposition, we first turn the diffeomorphism group $\mathrm{Diff}^\infty(B)$ of a ball $B \subseteq \mathbb{K}^d$ (where $d \in \mathbb{N}_0$) into a Lie group; this is quite easy, because $\mathrm{Diff}^\infty(B)$ is a mere open subset of $C^\infty(B, \mathbb{K}^d)$. Next, we form the weak direct product of Lie groups

$$\textstyle\prod^*_{i\in I} \mathrm{Diff}^\infty(B_i)$$

[1]Analogous results for the case where $k = r = \infty$, E is locally convex and both M and N are open subsets of real locally convex spaces (instead of manifolds) have been obtained earlier in [6], along with interesting additional information.

modeled on $\bigoplus_{i\in I} C^\infty(B_i, TB_i) = C^\infty_c(M, TM)$. This weak direct product can be identified with a subgroup of the group $\mathrm{Diff}^\infty_c(M)$ of "compactly supported" diffeomorphisms. It then only remains to equip $\mathrm{Diff}^\infty(M)$ with a smooth $\mathbb{K}$-manifold structure making it a Lie group with $\prod^*_{i\in I} \mathrm{Diff}^\infty(B_i)$ as an open subgroup. We remark that smoothness of the inversion map $\mathrm{Diff}^\infty(B) \to \mathrm{Diff}^\infty(B)$, $\gamma \mapsto \gamma^{-1}$ is a simple consequence of the exponential laws established here and a version of the implicit function theorem (the "Inverse Function Theorem with Parameters") for mappings from metrizable topological vector spaces to Banach spaces [28].

Second approach (Section 14). For our second approach to diffeomorphism groups, we assume that M is σ-compact and of positive dimension over $\mathbb{K}$. In this case, M is $C^\infty_{\mathbb{K}}$-diffeomorphic to an open subset U of its modeling space $\mathbb{K}^d$ (Lemma 8.3 (a); cf. [50]), making it quite easy to deal with M. Given $r \in \mathbb{N}\cup\{\infty\}$, we consider the monoid $\mathrm{End}^r_c(U)$ of all $C^r_{\mathbb{K}}$-maps $U \to U$ which coincide with id_U outside some compact set. We show that $\gamma \mapsto \gamma - \mathrm{id}_U$ identifies $\mathrm{End}^r_c(U)$ with an open subset of $C^r_c(U, \mathbb{K}^d)$. Thus $\mathrm{End}^r_c(U)$ is a $C^\infty_{\mathbb{K}}$-manifold with a global chart. We show that $\mathrm{Diff}^r_c(U) = \mathrm{End}^r_c(U)^\times$ is open in $\mathrm{End}^r_c(U)$, and we show that, for each $k \in \mathbb{N}_0 \cup \{\infty\}$, the composition map

$$\mathrm{Diff}^{r+k}_c(U) \times \mathrm{Diff}^r_c(U) \to \mathrm{Diff}^r_c(U)$$

and the inversion map $\mathrm{Diff}^{r+k}_c(U) \to \mathrm{Diff}^r_c(U)$ are $C^k_{\mathbb{K}}$. This enables us to turn $\mathrm{Diff}^\infty(M)$ into a Lie group with $\mathrm{Diff}^\infty_c(M) \cong \mathrm{Diff}^\infty_c(U)$ as an open subgroup, modeled on the space $C^\infty_c(M, TM)$ of compactly supported smooth vector fields on M, equipped with its natural LF vector topology.[2] But it also enables us to define a second Lie group structure on $\mathrm{Diff}^\infty(M)$ (which we then denote by $\mathrm{Diff}^\infty(M)\tilde{}$). It is modeled on the same vector space $C^\infty_c(M, TM)$, equipped however with the (in general properly) coarser vector topology making this space the projective limit

$$\bigcap_{k\in\mathbb{N}_0} C^k_c(M, TM) = \lim_{\longleftarrow\, k\in\mathbb{N}_0} C^k_c(M, TM)\,.$$

Apparently, the definition of this second Lie group structure is close in spirit to the ILB-approach to diffeomorphism groups in the works of Omori [68], [69]. An analogous construction for diffeomorphism groups of σ-compact real manifolds had been proposed in [58] (and was fully worked out in [33]). As in the real case, we can also give $\mathrm{Diff}^r(M)$ a smooth manifold structure for each finite r, with $\mathrm{Diff}^r_c(M) \cong \mathrm{Diff}^r_c(U)$ as an open subgroup, such that $\mathrm{Diff}^r(M)$ becomes a topological group and all right translations are smooth.

We remark that certain groups $\mathrm{Diff}(t, M)$, $G(t, M)$, and $GC(t, M)$ of diffeomorphisms of class of smoothness $C(t)$ for manifolds over local fields of characteristic zero have already been discussed in [49], [51], [55] and further works of S. V. Ludkovsky, where they are considered mainly as manifolds and topological groups (rather than Lie groups). He discusses irreducible representations of these groups ([51]) and measures on such groups (or

[2] The Lie group structure so obtained coincides with the one from Section 13.

M) which are quasi-invariant with respect to dense subgroups [49]. Ludkovsky's approach to differential calculus (which is necessarily restricted to local fields $\mathbb{K}$ of characteristic zero, and differs from the one we use), is described in [53, I, §2.3] (for maps between ultrametric Banach spaces) and extended to the case of locally convex topological $\mathbb{K}$-vector spaces in [53, II, Rem. 4.4] and [55], where diffeomorphism groups are discussed in further detail and generality. Ludkovsky also turns groups $\mathrm{Diff}^t_{\beta,\gamma}(M)$ of certain diffeomorphisms of real Banach manifolds (subject to Hölder-type conditions) into topological groups [52, Thm. 3.1], and discusses irreducible representations and quasi-invariant measures for such groups [52], [54]. Note that the "non-archimedian loop groups" discussed in [53] are not mapping groups in the sense considered in the present article, but something different.

Finally, let us mention that weak direct products of Lie groups are also useful to obtain information on diffeomorphism groups of real manifolds (although they are not simply open subgroups here, as in the ultrametric case). In [34], weak direct products are used to show that the Lie group $\mathrm{Diff}^\infty_c(M)$ of compactly supported diffeomorphisms of a σ-compact, finite-dimensional smooth manifold M is the direct limit $\varinjlim \mathrm{Diff}^\infty_K(M)$ both in the category of Lie groups modeled on real locally convex spaces and in the category of topological groups (where K ranges through the set of compact subsets of M, and $\mathrm{Diff}^\infty_K(M) := \{\gamma \in \mathrm{Diff}^\infty(M) \colon (\forall x \in M \setminus K)\ \gamma(x) = x\}$). This is remarkable, because, as a consequence of results from [31], in general the Lie group $\mathrm{Diff}^\infty_c(M)$ neither is the direct limit $\varinjlim \mathrm{Diff}^\infty_K(M)$ in the category of topological spaces, nor in the category of smooth manifolds ([34]; cf. also [76]). Analogous results can be obtained for the test functions groups $C^\infty_c(M, G)$, for G a finite-dimensional real Lie group [34].

Concluding remarks and guidance for the reader

Readers who wish to get quickly to the main results can skip part of the material. For example, since all of the vector bundles required for the discussion of diffeomorphism groups over local fields are trivial bundles, only very few of our results on spaces of sections in vector bundles (discussed in Appendix F) will actually be used, and these are easy to take on faith (cf. Remark 8.17). Proposition 4.16 (concerning pushforwards f_* for globally defined f) is only needed for the discussion of spaces of sections in vector bundles, while its more complicated variants (Propositions 4.20 and 4.23) are vital for the Lie group constructions. Nonetheless, the author recommends to study the proof of the simpler Proposition 4.16 first, and to leave the proofs of Propositions 4.20 and 4.23 for a second reading. Section 10 is only needed for our second approach to diffeomorphism groups (Section 14), but not for the first approach (Section 13). The general case of Proposition 11.3 (proved in Appendix C) is not used elsewhere, and only part of Section 12 (concerning the exponential law) is needed for the discussion of diffeomorphism groups: Lemma 12.1 (a), and Lemma 12.1 (b) for $k = 0$ suffice.

It is clear, however, that it would be inefficient not to include such closely related results, when this can be done without much additional effort. Besides their obvious potential for applications, the additional results also serve to put the Lie theoretic constructions in a

larger perspective, and thus foster their understanding.

Let us remark in closing that it was necessary to refrain from developing the surrounding theory up to the possible limits of generality, in order not to be carried away too far from the subject matter of Lie group constructions, to increase the readability, and to avoid the discussions from getting even more technical.

We mainly think of two possible generalizations. In the real locally convex case, a more refined discussion of the maps $C^r_c(M,f)\colon C^r_c(M,U) \to C^r_c(M,F)$ between open subsets of spaces of compactly supported sections is possible [32]; in this case, $C^r_c(M,f)$ is C^k provided, in local coordinates, f is C^k along the fibre, with fibre derivatives jointly C^r. An analogous condition, based on iterated partial difference quotient maps, should be sufficient to ensure that $C^r_c(M,f)$ be C^k, in the general case of bundles of topological vector spaces over finite-dimensional paracompact C^r-manifolds over locally compact fields. This would substantially generalize our version of the Ω-Lemma (which, however, already incorporates the cases of main relevance), but would inflict complicated technical arguments on us, which are irrelevant for the Lie group constructions.

The second possible generalization concerns the exponential law. If $k = r = \infty$, the map Φ from (2) should always be a topological embedding (cf. [6]). Furthermore, for general r and k, a more detailed analysis of the problem should reveal that Φ can be written as a composition

$$C^{r+k}(M \times N, E) \to C^{k,r}(M \times N, E) \to C^k(M, C^r(N,E))$$

of continuous linear injections for a suitably defined space $C^{k,r}(M \times N, E)$ of E-valued $C^{k,r}$-maps on $M \times N$. Here, the first mapping is the inclusion map. The second map, $C^{k,r}(M \times N, E) \ni f \mapsto f^\vee \in C^k(M, C^r(N,E))$, should always be a topological embedding. Again, the author feels that the immense additional technical effort would not be justified in the present context. The problems may be analyzed elsewhere.

1 Differential calculus over topological fields

It is possible to define C^k-mappings and smooth mappings once a topologized ring and a so-called C^0-concept is given, satisfying suitable axioms (see [3]). In the present paper, we exclusively consider the special case where the given topologized ring is a non-discrete topological field $\mathbb{K}$ (Hausdorff, as all topological spaces we consider), where C^0-maps are defined as continuous maps between open subsets of topological vector spaces over $\mathbb{K}$, and where the product topology is used on products of topological vector spaces. In this section, we briefly describe the resulting setting of differential calculus.

1.1 Conventions. All topological spaces occurring in this paper are assumed Hausdorff. All topological fields are supposed to be non-discrete. A field $\mathbb{K}$, together with an absolute value $|.|\colon \mathbb{K} \to [0,\infty[$ giving rise to a non-discrete topology on $\mathbb{K}$ will be called a *valued*

field. An *ultrametric field* is a valued field $(\mathbb{K}, |.|)$ whose absolute value is ultrametric, i.e.,

$$|x+y| \leq \max\{|x|, |y|\} \qquad \text{for all } x, y \in \mathbb{K}.$$

If $(\mathbb{K}, |.|)$ is an ultrametric field, then $\mathbb{O} := \{x \in \mathbb{K} : |x| \leq 1\}$ is a subring of $\mathbb{K}$, called the *valuation ring*. The valuation ring is an open and closed subset of $\mathbb{K}$. Totally disconnected, locally compact topological fields will be referred to as *local fields*. It is well known that every local field admits an ultrametric absolute value defining its topology, and can therefore be considered as a complete ultrametric field. The valuation ring of a local field is open and compact. It is also known that every locally compact topological field is either isomorphic to $\mathbb{R}$, $\mathbb{C}$, or a local field [78]. A complete[3] valued field $(\mathbb{K}, |.|_{\mathbb{K}})$ is either ultrametric or isomorphic as a valued field to $(\mathbb{R}, |.|^\alpha)$ or $(\mathbb{C}, |.|^\alpha)$ for some $\alpha \in \,]0, 1]$, where $|.|$ is the usual absolute value [11, VI, §6].

1.2 A topological vector space E over an ultrametric field $(\mathbb{K}, |.|)$ is called *locally convex* if every zero-neighbourhood of E contains an open $\mathbb{O}$-submodule of E, where $\mathbb{O}$ is the valuation ring of $\mathbb{K}$ (see [60], Ch. III, §2, Prop. 4 and §3, Déf. 1 when $\mathbb{K}$ is complete). It is well known that a topological vector space E over an ultrametric field is locally convex if and only if its topology arises from a family $(\|.\|_\gamma)_\gamma$ of *ultrametric* continuous seminorms $\|.\|_\gamma \colon E \to [0, \infty[$, satisfying $\|x+y\|_\gamma \leq \max\{\|x\|_\gamma, \|y\|_\gamma\}$ for all $x, y \in E$.

1.3 If $(E, \|.\|)$ is a normed space over a valued field $\mathbb{K}$, given $\varepsilon > 0$ and $x \in E$ we write $B^E_\varepsilon(x) := \{y \in E : \|y - x\| < \varepsilon\}$, or simply $B_\varepsilon(x) := B^E_\varepsilon(x)$ if E is understood. Note that, since the image of a norm $\|.\|$ need not be contained in the image $|\mathbb{K}|$ of the absolute value, it is not possible in general to normalize elements: Given $0 \neq x \in E$ we need not find $t \in \mathbb{K}^\times$ such that $\|tx\| = 1$. To ensure that $\|Ax\| \leq \|A\|\,\|x\|$, the operator norm of a linear operator $A : E \to F$ between normed spaces $(E, \|.\|_E)$ and $(F, \|.\|_F)$ therefore has to be defined as $\|A\| := \min\{C \geq 0 : (\forall x \in E)\ \|Ax\|_F \leq C\|x\|_E\}$. If $E = F = \mathbb{K}^d$ for some $d \in \mathbb{N}_0$ and $\|.\|_E = \|.\|_F$ is the maximum norm $\|.\|_\infty \colon \mathbb{K}^d \to [0, \infty[$, $\|(x_1, \ldots, x_d)\|_\infty := \max\{|x_1|, \ldots, |x_d|\}$, then every non-zero vector can be normalized and thus $\|A\| = \max\{\|Ax\|_\infty \colon x \in \mathbb{K}^d,\ \|x\|_\infty \leq 1\}$. As usual, given topological vector spaces E and F over a topological field $\mathbb{K}$, we let $\mathcal{L}(E, F)$ denote the set of all continuous linear maps from E to F; we abbreviate $\mathcal{L}(E) := \mathcal{L}(E, E)$.

Throughout the remainder of this section, $\mathbb{K}$ denotes a (non-discrete) topological field.

Before we define C^k-maps, we need an efficient notation for the domains of certain mappings associated with C^k-maps.

Definition 1.4 If E is a topological $\mathbb{K}$-vector space and $U \subseteq E$ an open subset, we define $U^{[0]} := U$ and

$$U^{[1]} := \{(x, y, t) \in U \times E \times \mathbb{K} \colon x + ty \in U\},$$

which is an open subset of the topological $\mathbb{K}$-vector space $E \times E \times \mathbb{K}$. Having defined $U^{[j]}$ inductively for a natural number $j \geq 1$, we set $U^{[j+1]} := (U^{[j]})^{[1]}$.

[3]The requirement is that $(\mathbb{K}, d)$ be a complete metric space, where $d \colon \mathbb{K} \times \mathbb{K} \to [0, \infty[$, $d(x, y) := |x - y|_{\mathbb{K}}$.

In particular, $E^{[1]} = E \times E \times \mathbb{K}$, $E^{[2]} = E \times E \times \mathbb{K} \times E \times E \times \mathbb{K} \times \mathbb{K}$, etc.

Definition 1.5 Let E and F be topological $\mathbb{K}$-vector spaces and $f\colon U \to F$ be a mapping, defined on an open subset $U \subseteq E$. We say that f is *of class* $C^0_{\mathbb{K}}$ if f is continuous, in which case we set $f^{[0]} := f$ and call $f^{[0]}$ the 0th extended difference quotient map of f. If f is continuous and there exists a continuous mapping $f^{[1]}\colon U^{[1]} \to F$ such that

$$\tfrac{1}{t}(f(x+ty) - f(x)) = f^{[1]}(x,y,t) \;\text{ for all } (x,y,t) \in U^{[1]} \text{ such that } t \neq 0, \tag{3}$$

we say that f is *of class* $C^1_{\mathbb{K}}$, and call $f^{[1]}$ the (first) *extended difference quotient map* of f (note that, as $\mathbb{K}$ is non-discrete, the continuous map $f^{[1]}$ is uniquely determined by (3)). Recursively, having defined $C^j_{\mathbb{K}}$-maps and jth extended difference quotient maps for $j = 0, \ldots, k-1$ for some natural number $k \geq 2$, we call f a mapping *of class* $C^k_{\mathbb{K}}$ if f is of class $C^{k-1}_{\mathbb{K}}$ and $f^{[k-1]}$ is of class $C^1_{\mathbb{K}}$. In this case, we define the kth extended difference quotient map of f via

$$f^{[k]} := (f^{[k-1]})^{[1]}\colon U^{[k]} \to F\,.$$

The mapping f is *of class* $C^\infty_{\mathbb{K}}$ (or $\mathbb{K}$-*smooth*) if it is of class $C^k_{\mathbb{K}}$ for all $k \in \mathbb{N}_0$. If $\mathbb{K}$ is understood, we simply write C^k instead of $C^k_{\mathbb{K}}$, and call f smooth or of class C^∞ if it is $\mathbb{K}$-smooth. If, conversely, we want to stress the fact that the field $\mathbb{K}$ is used, we shall write $U^{[k]}_{\mathbb{K}}$ for $U^{[k]}$ and $f^{[k]}_{\mathbb{K}}$ for $f^{[k]}$.

Examples 1.6 Every continuous $\mathbb{K}$-linear mapping $\lambda\colon E \to F$ between topological $\mathbb{K}$-vector spaces is smooth, with $\lambda^{[1]}(x,y,t) = \lambda(y)$ for all $(x,y,t) \in E \times E \times \mathbb{K}$. If V, W and F are topological $\mathbb{K}$-vector spaces and $\beta\colon V \times W \to F$ is a continuous bilinear map, then β is smooth, with

$$\beta^{[1]}((v,w),(v',w'),t) = \beta(v,w') + \beta(v',w) + t\beta(v',w')$$

for all $v, v' \in V$, $w, w' \in W$, and $t \in \mathbb{K}$ (cf. [3]).

1.7 Note that, for $k \geq 2$, a mapping f as above is of class $C^k_{\mathbb{K}}$ if and only if f is of class $C^1_{\mathbb{K}}$ and $f^{[1]}$ is of class $C^{k-1}_{\mathbb{K}}$; in this case, $f^{[k]} = (f^{[1]})^{[k-1]}$ (these claims are proved by a trivial induction).

1.8 Given a map $f\colon U \to F$ as before, let $f^{]1[}\colon U^{]1[} \to F$, $f^{]1[}(x,y,t) := \frac{1}{t}(f(x+ty) - f(x))$ be the associated difference quotient map, defined on $U^{]1[} := \{(x,y,t) \in U^{[1]}\colon t \neq 0\}$. Then f is $C^1_{\mathbb{K}}$ if and only if it is continuous and $f^{]1[}$ extends to a continuous function, $f^{[1]}$, on $U^{[1]}$. The set $U^{]1[}$ is open and dense in $U^{[1]}$, and we have $U^{[1]} = U^{]1[} \cup (U \times E \times \{0\})$, as a disjoint union. If f is $C^k_{\mathbb{K}}$, then so is $f^{]1[}$ (cf. **1.11** below).

1.9 Given a $C^1_{\mathbb{K}}$-map $f : U \to F$ as before, we define the *directional derivative* of f at $x \in U$ in the direction $v \in E$ via

$$df(x,v) \;:=\; (D_v f)(x) \;:=\; \lim_{0 \neq t \to 0} \tfrac{1}{t}(f(x+tv) - f(x)) \;=\; f^{[1]}(x,v,0)\,.$$

Then $df\colon U \times E \to F$ is continuous, being a partial map of $f^{[1]}$, and it can be shown that the "differential" $df(x,\bullet)\colon E \to F$ of f at x is a continuous $\mathbb{K}$-linear map, for each $x \in U$ [3, Prop. 2.2]. If f is C^2, we define a continuous map $d^2 f\colon U \times E^2 \to F$ via

$$\begin{aligned} d^2 f(x, v_1, v_2) &:= (D_{v_2}(D_{v_1} f))(x) = \lim_{0 \neq t \to 0} \tfrac{1}{t}(df(x + tv_2, v_1) - df(x, v_1)) \\ &= f^{[2]}(x, v_1, 0, v_2, 0, 0, 0). \end{aligned}$$

Similarly, if f is of class $C^k_{\mathbb{K}}$, we obtain continuous mappings (the "higher differentials") $d^j f : U \times E^j \to F$, $d^j f(x, v_1, \ldots, v_j) := (D_{v_j} \cdots D_{v_1} f)(x)$ for all $j \in \mathbb{N}_0$ such that $j \leq k$ (where $d^0 f := f$). Here $d^j f(x,\bullet)\colon E^j \to F$ is symmetric and j-multilinear [3, La. 4.8].

1.10 If $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ and the range F is locally convex, the considerations in **1.9** show that every $C^k_{\mathbb{K}}$-map in the preceding sense is a C^k-map in the sense of Michal-Bastiani (a C^k_{MB}-map for short), *i.e.*, the iterated directional derivatives needed to define $d^j f$ exist for all $j \in \mathbb{N}$ such that $j \leq k$, and the mappings $d^j f\colon U \times E^j \to F$ so obtained (as well as f) are continuous (such mappings are also called "Keller's C^k_c-maps" in the literature, following [44]). Exploiting the Fundamental Theorem of Calculus, it can be shown that, conversely, every C^k_{MB}-map with locally convex range is of class $C^k_{\mathbb{K}}$ [3, Prop. 7.4]. Thus, when dealing with maps into real or complex locally convex spaces, it is fully sufficient (and much more convenient) to work with the differentials $d^j f$, no use has to be made of the mappings $f^{[j]}$. However, as soon as we turn to mappings into non-locally convex real or complex topological vector spaces, and also in the case of base fields other than $\mathbb{R}$ and $\mathbb{C}$, the differentials alone do not encode enough information on f, and it is necessary to work with the mappings $f^{[j]}$. For example, consider the function $f : \,]0,1[\, \to L^0[0,1]$, $f(t) := \mathbf{1}_{[0,t]}$ taking t to the characteristic function of the interval $[0,t]$; here $[0,1]$ is equipped with Lebesgue measure, and $L^0[0,1]$ denotes the space of equivalence classes of measurable real-valued functions on $[0,1]$ (modulo equality a.e.), equipped with the topology of convergence in measure (see [42]). Then f is injective. It can be shown that f is of class $C^\infty_{\mathbb{R}}$, with $d^j f$ vanishing identically for all $j \in \mathbb{N}$ (cf. [24]).

1.11 (Chain Rule). If E, F and H are topological $\mathbb{K}$-vector spaces, $U \subseteq E$ and $V \subseteq F$ are open subsets, and $f\colon U \to V \subseteq F$, $g\colon V \to H$ are mappings of class C^k, then also the composition $g \circ f\colon U \to H$ is of class C^k. If $k \geq 1$, we have $(f(x), f^{[1]}(x,y,t), t) \in V^{[1]}$ for all $(x,y,t) \in U^{[1]}$, and

$$(g \circ f)^{[1]}(x, y, t) = g^{[1]}(f(x), f^{[1]}(x, y, t), t)\,. \tag{4}$$

In particular, $d(g \circ f)(x,y) = dg(f(x), df(x,y))$ for all $(x,y) \in U \times E$ (see [3], Prop. 3.1 and Prop. 4.5).

We recall that being of class C^k is a local property [3, La. 4.9]:

Lemma 1.12 *Let E and F be topological $\mathbb{K}$-vector spaces and $f : U \to F$ be a mapping, defined on an open subset U of E. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. If there is an open cover $(U_i)_{i \in I}$ of U such that $f|_{U_i}\colon U_i \to F$ is of class C^k for each $i \in I$, then f is of class C^k.* $\square$

1.13 Compositions of composable C^k-mappings being of class C^k, we can define C^k-manifolds modeled on topological $\mathbb{K}$-vector spaces in the usual way, namely as Hausdorff topological spaces M, together with a set ("atlas") of homeomorphisms ("charts") from open subsets of M onto open subsets of the modeling topological $\mathbb{K}$-vector space Z, whose domains cover M, and such that the transition maps are of class C^k. A *Lie group over* $\mathbb{K}$ is a group G, equipped with a smooth manifold structure modeled on a topological $\mathbb{K}$-vector space Z, with respect to which inversion and the group multiplication are smooth mappings. For every $\mathbb{K}$-Lie group G, the geometric tangent space $T_1(G)$ can be turned into a topological $\mathbb{K}$-Lie algebra $L(G)$ in a natural way (see [3] for more information).

Remark 1.14 It can be shown that $\mathbb{K}$-analytic maps from open subsets of (ultrametric) normed spaces to locally convex topological $\mathbb{K}$-vector spaces (as in [14], where locally convex spaces are called "polynormed") are $C^\infty_{\mathbb{K}}$, for every ultrametric field $(\mathbb{K}, |.|)$ [3, Prop. 7.20]. As a consequence, every finite-dimensional $\mathbb{K}$-analytic Lie group G in the usual sense (as defined in [74, p. 102]) can be considered as a $\mathbb{K}$-Lie group in our sense, and likewise for the analytic Lie groups modeled on ultrametric Banach spaces considered in [13].

We recall three simple, but very useful facts ([3], Lemmas 10.1–10.3):

Lemma 1.15 *Let E and F be topological $\mathbb{K}$-vector spaces, $U \subseteq E$ be open, and $f : U \to F$ be a mapping of class C^k, where $k \in \mathbb{N}_0 \cup \{\infty\}$. Let F_0 be a vector subspace of F containing the image of f. If F_0 is closed or if F_0 is sequentially closed and $\mathbb{K}$ is metrizable, then the co-restriction $f|^{F_0} : U \to F_0$ is C^k as a map into F_0.* □

Lemma 1.16 *Suppose that E is a topological $\mathbb{K}$-vector space, $(F_i)_{i\in I}$ a family of topological $\mathbb{K}$-vector spaces, $U \subseteq E$ an open subset, and $f : U \to P$ a mapping, where $P := \prod_{i\in I} F_i$, with canonical projections $\mathrm{pr}_i : P \to F_i$. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. Then f is of class C^k if and only if $\mathrm{pr}_i \circ f$ is of class C^k for each $i \in I$.* □

Lemma 1.17 *Let E be a topological $\mathbb{K}$-vector space, $(F_i)_{i\in I}$ be a family of topological $\mathbb{K}$-vector spaces, $U \subseteq E$ be open, and $f : U \to F$ be a map, where $F = \lim\limits_{\longleftarrow i\in I} F_i$, with limit maps $\pi_i : F \to F_i$. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. Then f is C^k iff $\pi_i \circ f$ is C^k for each $i \in I$.* □

As in the case of Banach-Lie groups, general Lie groups can be described locally:

Proposition 1.18 (Local description of Lie group structures) *Suppose that a subset U of a group G is equipped with a smooth manifold structure modeled on a topological $\mathbb{K}$-vector space E, and suppose that V is an open subset of U such that $1 \in V$, $V = V^{-1}$, $VV \subseteq U$, and such that the multiplication map $V \times V \to U$, $(g, h) \mapsto gh$ is smooth as well as inversion $V \to V$, $g \mapsto g^{-1}$; here V is considered as an open submanifold of U. Suppose that for every element x in a symmetric generating set of G, there is an open identity-neighbourhood $W \subseteq U$ such that $xWx^{-1} \subseteq U$, and such that the mapping $W \to U$, $w \mapsto xwx^{-1}$ is smooth.[4] Then there is a unique $\mathbb{K}$-Lie group structure on G which makes V, equipped with the above manifold structure, an open submanifold of G.*

Proof. The proof of [13], Chapter 3, §1.9, Proposition 18 can easily be adapted. □

[4]This condition is automatically satisfied if V generates G.

2 Unit groups of continuous inverse algebras

Let $\mathbb{K}$ be an arbitrary topological field. In this section, we show that the groups $A^\times$ of invertible elements in suitable topological $\mathbb{K}$-algebras A (the continuous inverse $\mathbb{K}$-algebras) are $\mathbb{K}$-Lie groups. We describe constructions producing new continuous inverse algebras from given ones. In this way, we obtain a supply of continuous inverse $\mathbb{K}$-algebras and thus also of $\mathbb{K}$-Lie groups. For much more information concerning locally convex continuous inverse algebras over the real or complex field, and their unit groups, the reader is referred to [20]. Further examples can be found in [23], [36, 1.15], and [81].

Definition 2.1 A *continuous inverse algebra* (over $\mathbb{K}$) is a unital associative topological $\mathbb{K}$-algebra A whose group of units $A^\times$ is open in A and such that inversion $\iota\colon A^\times \to A$, $a \mapsto a^{-1}$ is continuous.

Continuous inverse algebras are of interest in the present context, for the following reason:

Proposition 2.2 *For every continuous inverse $\mathbb{K}$-algebra A, inversion $\iota\colon A^\times \to A$ is of class $C^\infty_\mathbb{K}$, and thus $A^\times$ is a $\mathbb{K}$-Lie group when considered as an open submanifold of A.*

Proof. The algebra multiplication $A \times A \to A$ is continuous bilinear and hence smooth. Hence so is the group multiplication $A^\times \times A^\times \to A^\times$. We now show by induction that ι is $C^k_\mathbb{K}$ for each $k \in \mathbb{N}_0$. By hypothesis, ι is $C^0_\mathbb{K}$. Suppose that ι is $C^k_\mathbb{K}$. Using that $b^{-1} - a^{-1} = b^{-1}(a-b)a^{-1}$ for $a, b \in A^\times$, we obtain for any $(x,v,t) \in (A^\times)^{[1]} \subseteq A^\times \times A \times \mathbb{K}$:

$$\iota(x+tv) - \iota(x) = (x+tv)^{-1} - x^{-1} = -t((x+tv)^{-1}vx^{-1} = tF(x,v,t)\,, \tag{5}$$

where $F : (A^\times)^{[1]} \to A$, $F(x,v,t) := -\iota(x+tv)v\iota(x)$. Since ι is $C^k_\mathbb{K}$ by the induction hypotheses, F is of class $C^k_\mathbb{K}$, in particular of class $C^0_\mathbb{K}$. Thus (5) shows that ι is of class $C^1_\mathbb{K}$, with $\iota^{[1]} = F$ a mapping of class $C^k_\mathbb{K}$. Therefore ι is of class $C^{k+1}_\mathbb{K}$ (see **1.7**). □

For example, $\mathbb{K}$ is a continuous inverse algebra over $\mathbb{K}$, and thus $\mathbb{K}^\times$ is a $\mathbb{K}$-Lie group.

Proposition 2.3 *If A is a continuous inverse $\mathbb{K}$-algebra, then so is the algebra $M_n(A)$ of $n\times n$-matrices with entries in A, when equipped with the natural vector topology $\cong A^{n\times n}$.*

Proof. Apparently, $M_n(A)$ is a topological $\mathbb{K}$-algebra. Its unit group is open and inversion is continuous by [75, Cor. 1.2]. □

Lemma 2.4 *If A is a finite-dimensional unital associative $\mathbb{K}$-algebra and $B \subseteq A$ a unital subalgebra, then $B^\times = A^\times \cap B$.*

Proof. This is a well-known fact (cf. [20, La. 9.4] if $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). □

2.5 Recall that, among the vector topologies on a finite-dimensional $\mathbb{K}$-vector space V of dimension d, there is a uniquely determined vector topology making V isomorphic to the direct product $\mathbb{K}^d$ as a topological vector space. It is called the *canonical $\mathbb{K}$-vector space topology* (see [12], Ch. I, §1, no. 1, Example 5).

We remark that it is possible to characterize those topological fields $\mathbb{K}$ having the special property that any finite-dimensional $\mathbb{K}$-vector space admits only one (Hausdorff) vector topology ([62], also [79], Section 5.4, Theorem 10). They are necessarily complete.

Proposition 2.6 *Let A be a finite-dimensional unital associative $\mathbb{K}$-algebra. Then the canonical $\mathbb{K}$-vector space topology turns A into a continuous inverse algebra over $\mathbb{K}$.*

Proof. Let $n := \dim_{\mathbb{K}}(A)$. It is clear that the canonical $\mathbb{K}$-vector space topology turns A into a topological $\mathbb{K}$-algebra, and it is clear that the left regular representation

$$\lambda \colon A \to \mathcal{L}(A) \cong M_n(\mathbb{K}), \quad \lambda(a)(b) := ab$$

is a topological embedding (where $\mathcal{L}(A)$ denotes the $\mathbb{K}$-algebra of $\mathbb{K}$-linear self-maps of A, equipped with the canonical $\mathbb{K}$-vector space topology). It therefore suffices to assume that A is a subalgebra of $M_n(\mathbb{K})$. Now, $M_n(\mathbb{K})$ being a finite-dimensional $\mathbb{K}$-algebra, we have

$$A^\times = M_n(\mathbb{K})^\times \cap A \tag{6}$$

by Lemma 2.4. Since $M_n(\mathbb{K})$ is a continuous inverse algebra by Proposition 2.3, $M_n(\mathbb{K})^\times$ is open in $M_n(\mathbb{K})$ and thus $A^\times$ is open in A by (6). The inversion map $\iota : A^\times \to A$ being a restriction of the continuous inversion map $M_n(\mathbb{K})^\times \to M_n(\mathbb{K})$, we deduce that ι is continuous. □

Tensor products of finite-dimensional algebras and continuous inverse algebras are again continuous inverse algebras.

Proposition 2.7 *Given a continuous inverse $\mathbb{K}$-algebra A and finite-dimensional unital associative $\mathbb{K}$-algebra F, consider the associative unital $\mathbb{K}$-algebra $F \otimes_{\mathbb{K}} A$. Pick any $\mathbb{K}$-basis $e_1, \ldots, e_n$ of F, and equip $F \otimes_{\mathbb{K}} A$ with the topology making $\phi : A^n \to F \otimes_{\mathbb{K}} A$, $(a_i)_{i=1}^n \mapsto \sum_{i=1}^n e_i \otimes a_i$ an isomorphism of topological $\mathbb{K}$-vector spaces. Then this topology does not depend on the choice of basis, and it turns $F \otimes_{\mathbb{K}} A$ into a continuous inverse algebra over $\mathbb{K}$.*

Proof. The natural map $M_n(\mathbb{K}) \times A^n \to A^n$ being continuous, we readily deduce that the topology on $F \otimes_{\mathbb{K}} A$ is independent of the choice of $\mathbb{K}$-basis for F. Given $i, j \in \{1, \ldots, n\}$, we have $e_i e_j = \sum_{k=1}^n t_{i,j,k} e_k$ for uniquely determined elements ("structure constants") $t_{i,j,k} \in \mathbb{K}$. Given $z = (z_i)_{i=1}^n$, $v = (v_i)_{i=1}^n$ in A^n, we have

$$\phi(z) \cdot \phi(v) = \sum_{k=1}^n e_k \otimes \left(\sum_{i,j=1}^n t_{i,j,k} z_i v_j \right) = \phi \left(\Big(\sum_{i,j=1}^n t_{i,j,k} z_i v_j \Big)_{k=1}^n \right) .$$

As A is a topological $\mathbb{K}$-algebra, we readily deduce from the preceding formula that multiplication in $F \otimes_{\mathbb{K}} A$ is continuous. Thus $F \otimes_{\mathbb{K}} A$ is a topological $\mathbb{K}$-algebra. Given z and v as before, we calculate

$$\begin{aligned}
(1+\phi(z))\cdot(1+\phi(v)) &= (1+\sum_{i=1}^{n} e_i \otimes z_i)\cdot(1+\sum_{j=1}^{n} e_j \otimes v_j) \\
&= 1+\sum_{i,j=1}^{n} \underbrace{e_i e_j}_{=\sum_{k=1}^{n} t_{i,j,k} e_k} \otimes z_i v_j + \sum_{k=1}^{n} e_k \otimes z_k + \sum_{k=1}^{n} e_k \otimes v_k \\
&= 1+\sum_{k=1}^{n} e_k \otimes (z_k + (S(z).v)_k)\,, \qquad (7)
\end{aligned}$$

where $S(z) := (a_{kj}(z))_{k,j=1}^{n} \in M_n(A)$ with $a_{kj}(z) := \delta_{k,j} + \sum_{i=1}^{n} z_i t_{i,j,k}$, and where $(S(z).v)_k$ denotes the kth coordinate of the vector $S(z).v \in A^n$. We strive to show that for $z \in A^n$ in some zero-neighbourhood, we can choose v such that $S(z).v = -z$. Then, by Equation (7), the element $1+\phi(v)$ will be a right inverse for $1+\phi(z)$.

Since $M_n(A)$ is a continuous inverse $\mathbb{K}$-algebra (Proposition 2.3), and $S\colon A^n \to M_n(A)$ is a continuous mapping such that $S(0) = \mathbf{1} \in M_n(A)^\times$, there is a zero-neighbourhood U in A^n such that $S(U) \subseteq M_n(A)^\times$. Apparently, the mapping

$$\rho\colon U \to A^n, \quad \rho(z) := -S(z)^{-1}.z$$

is continuous. For each $z \in U$, we have $S(z)\rho(z) = -z$, and thus $(1+\phi(z))\cdot(1+\phi(\rho(z))) = 1$ by (7). Thus, for each a in the open identity neighbourhood $V := 1+\phi(U) \subseteq F \otimes_{\mathbb{K}} A$, the element

$$r(a) := 1+\phi(\rho(\phi^{-1}(a-1))) \in F \otimes_{\mathbb{K}} A$$

is a right inverse for a in $F \otimes_{\mathbb{K}} A$, and the mapping $r : V \to F \otimes_{\mathbb{K}} A$ is continuous. Very similar arguments show that there is an identity neighbourhood W in $F \otimes_{\mathbb{K}} A$ such that every $a \in W$ has a left inverse in $F \otimes_{\mathbb{K}} A$. Then $P := V \cap W$ is an identity neighbourhood in $F \otimes_{\mathbb{K}} A$ such that $P \subseteq (F \otimes_{\mathbb{K}} A)^\times$, and the inversion map $\iota : (F \otimes_{\mathbb{K}} A)^\times \to F \otimes_{\mathbb{K}} A$ satisfies $\iota|_P = r|_P$ an thus is continuous on P. As a consequence, $(F \otimes_{\mathbb{K}} A)^\times$ is open in $F \otimes_{\mathbb{K}} A$ and ι is continuous (cf. [20, La. 2.8]). □

Concerning extension of scalars, we readily deduce:

Corollary 2.8 *For every continuous inverse algebra A over $\mathbb{K}$ and finite extension field $\mathbb{L}$ of $\mathbb{K}$, $A_{\mathbb{L}} := \mathbb{L} \otimes_{\mathbb{K}} A$ is a continuous inverse algebra over $\mathbb{L}$ (where $\mathbb{L}$ is equipped with the canonical $\mathbb{K}$-vector space topology).*

Proof. We equip $A_{\mathbb{L}} = \mathbb{L} \otimes_{\mathbb{K}} A$ with the topological $\mathbb{K}$-algebra structure defined in Proposition 2.7, which makes it a continuous inverse algebra over $\mathbb{K}$. It is easy to see that the mapping $\mathbb{L} \to \mathbb{L} \otimes_{\mathbb{K}} A$, $z \mapsto z \otimes 1$ is a continuous $\mathbb{K}$-algebra homomorphism. The

continuity of scalar multiplication $\mathbb{L} \times A_{\mathbb{L}} \to A_{\mathbb{L}}$ therefore follows from the continuity of the multiplication map $A_{\mathbb{L}} \times A_{\mathbb{L}} \to A_{\mathbb{L}}$. $\square$

For the final result of this section, we specialize to the case where $\mathbb{K}$ is a locally compact topological field.

2.9 We consider a unital associative $\mathbb{K}$-algebra A which is *locally finite* in the sense that every finite subset is contained in a finite-dimensional subalgebra of A. We also assume that A is of countable dimension as a $\mathbb{K}$-vector space. As a consequence, there exists an ascending sequence $A_1 \subseteq A_2 \subseteq \cdots$ of finite-dimensional unital subalgebras A_n of A such that $A = \bigcup_{n\in\mathbb{N}} A_n$.

2.10 We equip A with the so-called "finite topology," *i.e.*, the final topology with respect to the inclusion maps $\lambda_F \colon F \to A$, where F runs through the set $\mathcal{F}$ of finite-dimensional vector subspaces of A. The set $\{A_n : n \in \mathbb{N}\}$ being co-final in $\mathcal{F}$ (directed with respect to inclusion), the finite topology on A is also the final topology with respect to the family $(\lambda_{A_n})_{n\in\mathbb{N}}$. Then $A = \varinjlim A_n$ as a topological space, furthermore $A \times A = \varinjlim (A_n \times A_n)$ and $\mathbb{K} \times A = \varinjlim (\mathbb{K} \times A_n)$, each A_n and $\mathbb{K}$ being locally compact ([40] or [21, Prop. 3.3]). As a consequence, A is a topological $\mathbb{K}$-algebra (cf. [21]).

Proposition 2.11 *Every countable-dimensional, locally finite associative unital algebra over a locally compact topological field* $\mathbb{K}$ *is a continuous inverse* $\mathbb{K}$*-algebra when equipped with the finite topology.*

Proof. We have already shown that A is a topological $\mathbb{K}$-algebra. The openness of $A^\times$ in A as well as continuity of inversion can be shown as in the real and complex special cases (see [20, Prop. 9.5]). $\square$

Remark 2.12 If A is a real or complex locally convex CIA, then $A^\times$ is in fact an analytic Lie group. If, furthermore, A is complete (or, at least, Mackey complete), then $A^\times$ is a Baker-Campbell-Hausdorff (BCH) Lie group, *viz.* it possesses a locally analytically diffeomorphic exponential function, and its multiplication is given locally by the BCH-series (see [20]). In this case, the results of [71] and [19] facilitate to integrate closed Lie subalgebras of A to analytic subgroups of $A^\times$, providing us with a much richer supply of "linear Lie groups" then the mere full unit groups $A^\times$. In the case where $\mathbb{K}$ is a complete valued field, in some cases subgroups of unit groups of continuous inverse $\mathbb{K}$-algebras may be turned into Lie groups using the inverse function and implicit function theorems from [28].

3 Spaces of continuous mappings and mappings between them

As a preliminary for our studies in Section 5, where we shall turn the group $C(K, G)$ of continuous mappings from a compact topological space K to a $\mathbb{K}$-Lie group G into a

$\mathbb{K}$-Lie group, in the present section we study differentiability properties of certain types of mappings between spaces of continuous vector-valued functions on compact topological spaces. More generally, here (and in Section 5) we can consider mappings on non-compact spaces supported in given compact sets.

Throughout this section, $\mathbb{K}$ denotes a topological field, X a topological space, and K a compact subset of X.

3.1 If E is a topological $\mathbb{K}$-vector space, we let $C_K(X,E) \subseteq E^X$ denote the $\mathbb{K}$-vector space of all continuous mappings $\gamma\colon X \to E$ such that $\operatorname{supp}(\gamma) \subseteq K$. We equip $C_K(X,E)$ with the topology of uniform convergence, which apparently makes $C_K(X,E)$ a topological $\mathbb{K}$-vector space. A basis of open zero-neighbourhoods is given by the sets $C_K(X,U) := \{\gamma \in C_K(X,E)\colon \operatorname{im}\gamma \subseteq U\}$, where U ranges through the open zero-neighbourhoods in E.

3.2 Note that if $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and E is locally convex, then $C_K(X,U)$ is convex for each convex, open 0-neighbourhood $U \subseteq E$ and thus $C_K(X,E)$ is locally convex. If $\mathbb{K}$ is an ultrametric field with valuation ring $\mathbb{O}$ and E is locally convex (see **1.2**), then $C_K(X,U)$ is an open $\mathbb{O}$-submodule of $C_K(X,E)$ for each open $\mathbb{O}$-submodule $U \subseteq E$, and hence $C_K(X,E)$ is locally convex.

The following proposition (and a C^r-analogue to be proved later) is the technical backbone of our discussion of mapping groups.

Proposition 3.3 *Let E, F, and Z be topological $\mathbb{K}$-vector spaces, $U \subseteq E$ and $P \subseteq Z$ be open subsets, $k \in \mathbb{N}_0 \cup \{\infty\}$, and $f\colon X \times U \times P \to F$ be a mapping. Suppose that*

(a) $f(x, \bullet) = 0$ *for all* $x \in X \setminus K$;

(b) $f(x, \bullet)\colon U \times P \to F$ *is of class* C^k *for each* $x \in X$, *and*

(c) $X \times (U \times P)^{[j]} \to F$, $(x,y) \mapsto f(x,\bullet)^{[j]}(y)$ *is a continuous map, for each* $j \in \mathbb{N}_0$ *such that* $j \leq k$.

Then $C_K(X,U) := C_K(X,E) \cap U^X$ is a (possibly empty) open subset of $C_K(X,E)$, and

$$\phi\colon C_K(X,U) \times P \to C_K(X,F), \quad \phi(\gamma,p) := f(\bullet,p)_*(\gamma)$$

is a mapping of class C^k (where $f(\bullet,p)_*(\gamma)(x) := f(x,\gamma(x),p)$ for $x \in X$).

Proof. It is clear that $C_K(X,U)$ is open, and that $\phi(\gamma,p) \in C_K(X,F)$ indeed. To show that ϕ is of class C^k, we clearly may assume that $k < \infty$. The proof is by induction.

The case $k = 0$. Let $\xi \in C_K(X,U)$, $p \in P$, and $V \subseteq F$ be an open zero-neighbourhood. Let $W \subseteq F$ be an open zero-neighbourhood such that $W - W \subseteq V$. For each $x \in K$, we find an open neighbourhood $A_x \subseteq K$ of x in K and open zero-neighbourhoods $B_x \subseteq E$ and $C_x \subseteq Z$ such that $\xi(A_x) + B_x \subseteq U$, $p + C_x \subseteq P$, and

$$f(y,u,q) - f(x,\xi(x),p) \in W$$

for all $y \in A_x$, $u \in \xi(A_x) + B_x$, and $q \in p + C_x$. By compactness, $K \subseteq \bigcup_{x\in I} A_x$ for some finite subset $I \subseteq K$. Then $B := \bigcap_{x\in I} B_x \subseteq E$ and $C := \bigcap_{x\in I} C_x \subseteq Z$ are open zero-neighbourhoods. Let $\eta \in \xi + C_K(X,B)$ and $q \in p + C \subseteq P$. Given $y \in K$, there is $x \in I$ such that $y \in A_x$. Hence

$$\begin{aligned} f(y,\eta(y),q) - f(y,\xi(y),p) &= f(y,\eta(y),q) - f(x,\xi(x),p) - (f(y,\xi(y),p) - f(x,\xi(x),p)) \\ &\in W - W \subseteq V. \end{aligned}$$

For $y \in X \setminus K$ on the other hand, we have $f(y,\eta(y),q) = f(y,\xi(y),p) = 0$ and thus $f(y,\eta(y),q) - f(y,\xi(y),p) = 0 \in V$ trivially. We have shown that $\phi(\eta,q) - \phi(\xi,p) \in C_K(X,V)$ for all (η,q) in the open neighbourhood $(\xi + C_K(X,B)) \times (p+C)$ of (ξ,p). Thus ϕ is continuous, as required.

Induction step. Suppose that $k \geq 1$, and suppose that the proposition holds for k replaced with $k-1$. Abbreviate $Q := (C_K(X,U) \times P)^{[1]} \subseteq C_K(X,U) \times P \times C_K(X,E) \times Z \times \mathbb{K}$ and $Q^\times := \{(\xi,p,\eta,q,t) \in Q \colon t \neq 0\}$. For all $(\xi,p,\eta,q,t) \in Q^\times$, we have

$$\begin{aligned} \tfrac{1}{t}(\phi(\xi + t\eta, p + tq) - \phi(\xi,p))(x) &= \tfrac{1}{t}(f(x,\xi(x) + t\eta(x), p + tq) - f(x,\xi(x),p)) \\ &= f(x,\bullet)^{[1]}((\xi(x),p),\ (\eta(x),q),\ t) \end{aligned} \tag{8}$$

for all $x \in X$, which suggests to define

$$\psi\colon Q \to C_K(X,F), \quad \psi(\xi,p,\eta,q,t)(x) := f(x,\bullet)^{[1]}((\xi(x),p),\ (\eta(x),q),\ t) \quad \text{for } x \in X.$$

If we can show that ψ is continuous, then ϕ will be C^1 with $\phi^{[1]} = \psi$, by (8).

Claim 1. *ψ is of class C^{k-1} on $Q^\times$.* In fact, inversion $\mathbb{K}^\times \to \mathbb{K}^\times$ being smooth, addition and scalar multiplication in $C_K(X,E)$ and $C_K(X,F)$ being continuous linear (resp., bilinear) and thus smooth, and ϕ being of class C^{k-1} by induction, the claim readily follows from the formula $\psi(\xi,p,\eta,q,t) = \frac{1}{t}(\phi(\xi + t\eta, p + tq) - \phi(\xi,p))$ for $(\xi,p,\eta,q,t) \in Q^\times$.

Claim 2. *Every $(\xi,p,\eta,q,0) \in Q$ has an open neighbourhood on which ψ is of class C^{k-1}.* In fact, since $\operatorname{im}\xi \subseteq \xi(K) \cup \{0\}$ and $\operatorname{im}\eta \subseteq \eta(K) \cup \{0\}$ are compact subsets of E, and $\operatorname{im}\xi \subseteq U$, there exist open neighbourhoods $A \subseteq U$ of $\operatorname{im}\xi$, $B \subseteq E$ of $\operatorname{im}\eta$ and an open zero-neighbourhood $C \subseteq \mathbb{K}$ such that $A + C \cdot B \subseteq U$. Shrinking C if necessary, we may furthermore assume that there exist open neighbourhoods $D \subseteq P$ of p and $G \subseteq Z$ of q such that $D + C \cdot G \subseteq P$. Then $U_1 := A \times B$ is an open subset of $E \times E$ containing $\operatorname{im}(\xi,\eta)$, and $P_1 := D \times G \times C$ is an open neighbourhood of $(p,q,0)$ in $P \times Z \times \mathbb{K}$. The definition

$$f_1\colon X \times U_1 \times P_1 \to F, \quad f_1(x,(a,b),(p',q',t)) := f(x,\bullet)^{[1]}((a,p'),(b,q'),t)$$

makes sense by choice of U_1 and P_1 (*i.e.*, the expression on the right hand side is defined). As an immediate consequence of hypothesis (c), the mapping $f_1(x,\bullet)$ is of class C^{k-1}, for each $x \in X$, and $X \times (U_1 \times P_1)^{[j]} \to F$, $(x,y) \mapsto f_1(x,\bullet)^{[j]}(y)$ is continuous for all $j \in \mathbb{N}_0$ such that $j \leq k-1$. By induction,

$$\phi_1\colon C_K(X,U_1) \times P_1 \to C_K(X,F), \quad \phi_1(\gamma,p_1)(x) := f_1(x,\gamma(x),p_1)$$

is a mapping of class C^{k-1}. Since $\phi_1((\sigma,\tau),(p',q',t)) = \psi(\sigma,p',\tau,q',t)$ for all $(\sigma,p',\tau,q',t) \in C_K(X,A) \times D \times C_K(X,B) \times G \times C \subseteq Q$, Claim 2 is established.

In view of Claims 1 and 2, Lemma 1.12 shows that ψ is a mapping of class C^{k-1}. In particular, ψ is continuous and thus, in view of (8), the mapping ϕ is of class C^1 with $\phi^{[1]} = \psi$ of class C^{k-1}. Thus ϕ is of class C^k. □

We readily deduce:

Corollary 3.4 *Let E and F be topological $\mathbb{K}$-vector spaces, $U \subseteq E$ an open subset, $k \in \mathbb{N}_0 \cup \{\infty\}$, and $f \colon X \times U \to F$ be a mapping. Suppose that*

(a) *$f(x,\bullet) = 0$ for all $x \in X \setminus K$;*

(b) *$f(x,\bullet)\colon U \to F$ is of class C^k for each $x \in X$, and*

(c) *$X \times U^{[j]} \to F$, $(x,y) \mapsto f(x,\bullet)^{[j]}(y)$ is continuous, for each $j \in \mathbb{N}_0$ with $j \leq k$.*

Then

$$f_* \colon C_K(X,U) \to C_K(X,F), \quad f_*(\gamma)(x) := f(x,\gamma(x))$$

is a mapping of class C^k. □

Corollary 3.5 *Let E and F be topological $\mathbb{K}$-vector spaces and $f \colon U \to F$ be a mapping of class C^k, defined on an open subset U of E. If $K \neq X$, assume that $0 \in U$ and $f(0) = 0$. Then*

$$C_K(X,f)\colon C_K(X,U) \to C_K(X,F), \quad \gamma \mapsto f \circ \gamma$$

is a mapping of class C^k.

Proof. We have $C_K(X,f) = g_*$, where $g \colon X \times U \to F$, $g(x,y) := f(y)$ is easily seen to satisfy Conditions (a), (b), (c) of Corollary 3.4. □

Before working through the analogues of the preceding facts for spaces of C^r-maps stated in the next section—which are considerably harder to prove—the reader may wish to pass directly to the construction of continuous mapping groups in Section 5 (assuming $r = 0$ there), to see what the results just proved are good for.

4 Spaces of C^r-maps and mappings between them

In this section, we discuss spaces of vector-valued C^r-maps, and mappings between such spaces, to facilitate the construction of a manifold structure on groups of C^r-maps in Section 5. We begin with the special case of vector-valued mappings on open subsets of topological vector spaces.

In this section, $\mathbb{F}$ denotes a topological field, and $\mathbb{K}$ a topological field extending $\mathbb{F}$, meaning

that $\mathbb{K}$ contains $\mathbb{F}$ as a subfield, and that the inclusion map $\mathbb{F} \to \mathbb{K}$ is continuous.[5] Starting with Proposition 4.20, we shall assume that $\mathbb{F}$ is locally compact. We remark that, if $\mathbb{F}$ is a valued field (for instance, if $\mathbb{F}$ is locally compact), then the inclusion map $\mathbb{F} \to \mathbb{K}$ is a topological embedding automatically, as every 1-dimensional (Hausdorff) topological $\mathbb{F}$-vector space is topologically isomorphic to $\mathbb{F}$ ([79], §5.1, Example 1 and §5.4, Theorem 9).

The spaces $C^r(U,E)$, when U is an open subset of the modeling space

The preparatory results concerning mappings on open subsets U of topological vector spaces provided in this subsection are essential for our later discussion of the general case, where U is replaced with a manifold.

4.1 Given a topological $\mathbb{K}$-vector space E and open subset U of a topological $\mathbb{F}$-vector space Z, we let $C^r(U,E)$ be the set of all mappings $\gamma : U \to E$ of class $C^r_{\mathbb{F}}$ (where $r \in \mathbb{N}_0 \cup \{\infty\}$). It is clear that pointwise operations turn $C^r(U,E)$ into a $\mathbb{K}$-vector space. We give $C^r(U,E)$ the initial topology with respect to the family of mappings

$$C^r(U,E) \to C(U^{[j]},E), \quad \gamma \mapsto \gamma^{[j]},$$

where $j \in \mathbb{N}_0$ such that $j \leq r$, and where $C(U^{[j]},E)$ is equipped with the topology of uniform convergence on compact sets (which coincides with the compact-open topology). It is clear that $C^r(U,E)$ becomes a topological $\mathbb{K}$-vector space in this way.

The sets

$$\lfloor K,W \rfloor := \{\gamma \in C(U^{[j]},E) \colon \gamma(K) \subseteq W\}$$

form a basis of open zero-neighbourhoods for the topology on $C(U^{[j]},E)$ when K ranges through the compact subsets of $U^{[j]}$ and W through the open zero-neighbourhoods of E.

Remark 4.2 The following assertions readily follow from the definitions:

(a) For every $r \geq s$, the inclusion map $C^r(U,E) \to C^s(U,E)$ is a continuous linear map. The topology on $C^\infty(U,E)$ is initial with respect to the family of inclusion maps $C^\infty(U,E) \to C^k(U,E)$, where $k \in \mathbb{N}_0$. Furthermore, $C^\infty(U,E) = \varprojlim C^k(U,E)$.

(b) For every $k \in \mathbb{N}_0$, the topology on $C^{k+1}(U,E)$ is initial with respect to the inclusion map $C^{k+1}(U,E) \to C(U,E)$ together with the mapping

$$C^{k+1}(U,E) \to C^k(U^{[1]},E), \quad \gamma \mapsto \gamma^{[1]}.$$

Strengthening (b), we have:

Lemma 4.3 *In the preceding situation, the map*

$$\Lambda : C^{k+1}(U,E) \to C(U,E) \times C^k(U^{[1]},E), \quad \Lambda(\gamma) := (\gamma, \gamma^{[1]})$$

is a topological embedding onto a closed vector subspace of $C(U,E) \times C^k(U^{[1]},E)$.

[5] Typical examples are: 1. $\mathbb{K} = \mathbb{F}$; 2. $\mathbb{F} = \mathbb{R}$, $\mathbb{K} = \mathbb{C}$.

Proof. By Remark 4.2 (b), the map Λ is a topological embedding. To see that $\operatorname{im}(\Lambda)$ is closed, let (γ_α) be a net in $C^{k+1}(U,E)$ such that $\Lambda(\gamma_\alpha)$ converges in $C(U,E)\times C^k(U^{[1]},E)$, say to (γ,η) with $\gamma\in C(U,E)$ and $\eta\in C^k(U^{[1]},E)$. Let $(x,y,t)\in U^{]1[}$ (see **1.8** for the notation). Then $\gamma_\alpha(x)\to\gamma(x)$ and $\gamma_\alpha(x+ty)\to\gamma(x+ty)$, entailing that $(\gamma_\alpha)^{[1]}(x,y,t)=\frac{1}{t}(\gamma_\alpha(x+ty)-\gamma_\alpha(x))\to\frac{1}{t}(\gamma(x+ty)-\gamma(x))$. Since also $(\gamma_\alpha)^{[1]}(x,y,t)\to\eta(x,y,t)$, we deduce that $\eta(x,y,t)=\frac{1}{t}(\gamma(x+ty)-\gamma(x))$. The map η being continuous, this means that γ is $C^1_{\mathbb{F}}$, with $\gamma^{[1]}=\eta$. The map $\gamma^{[1]}=\eta$ being $C^k_{\mathbb{F}}$, we deduce that γ is $C^{k+1}_{\mathbb{F}}$ and thus $\gamma\in C^{k+1}(U,E)$. Then $\lim\Lambda(\gamma_\alpha)=(\gamma,\eta)=(\gamma,\gamma^{[1]})=\Lambda(\gamma)$. Thus $\operatorname{im}(\Lambda)$ is closed. □

Lemma 4.4 *Suppose that Z and Y are topological $\mathbb{F}$-vector spaces, $U\subseteq Z$ and $V\subseteq Y$ open subsets, and $f\colon U\to V$ a $C^r_{\mathbb{F}}$-map. Then the "pullback"*

$$C^r(f,E)\colon C^r(V,E)\to C^r(U,E),\quad \gamma\mapsto\gamma\circ f$$

is a continuous $\mathbb{K}$-linear map.

Proof. Given elements $r\geq s\in\mathbb{N}_0\cup\{\infty\}$, let $i_{s,r}:C^r(U,E)\to C^s(U,E)$ and $j_{s,r}:C^r(V,E)\to C^s(V,E)$ be the respective inclusion maps. Since $i_{k,\infty}\circ C^\infty(f,E)=C^k(f,E)\circ j_{k,\infty}$ if f is of class $C^\infty_{\mathbb{F}}$, in view of Remark 4.2 (a) the continuity of $C^\infty(f,E)$ follows if we can show that $C^k(f,E)$ is continuous for each $k\in\mathbb{N}_0$. Thus, we may assume that $r\in\mathbb{N}_0$, and prove the assertion by induction on r. The case $r=0$ is a standard fact, see [16], p. 157, Assertion (2).

Induction step. Suppose the lemma is correct for some $r\in\mathbb{N}_0$, and suppose that $f\colon U\to V$ is of class $C^{r+1}_{\mathbb{F}}$. The mapping $i_{0,r+1}\circ C^{r+1}(f,E)=C^0(f,E)\circ j_{0,r+1}$ being continuous, in view of Remark 4.2 (b) it only remains to show that

$$\phi\colon C^{r+1}(V,E)\to C^r(U^{[1]},E),\quad \phi(\gamma):=\big(C^{r+1}(f,E)(\gamma)\big)^{[1]}=(\gamma\circ f)^{[1]}$$

is continuous. By the Chain Rule, we have

$$\phi(\gamma)=(\gamma\circ f)^{[1]}=C^r(\Phi,E)(\gamma^{[1]}),$$

where $\Phi:U^{[1]}\to V^{[1]}$, $\Phi(u,y,t):=(f(u),f^{[1]}(u,y,t),t)$ is of class C^r. By induction, $C^r(\Phi,E)\colon C^r(V^{[1]},E)\to C^r(U^{[1]},E)$ is continuous, and also $\psi\colon C^{r+1}(V,E)\to C^r(V^{[1]},E)$, $\gamma\mapsto\gamma^{[1]}$ is continuous (Remark 4.2 (b)). Thus $\phi=C^r(\Phi,E)\circ\psi$ is continuous. □

Lemma 4.5 *Let E be a topological $\mathbb{K}$-vector space, Z a topological $\mathbb{F}$-vector space, $U\subseteq Z$ an open subset, and $f:U\to\mathbb{K}$ be a mapping of class $C^r_{\mathbb{F}}$. Then the "multiplication operator"*

$$m_f\colon C^r(U,E)\to C^r(U,E),\quad (m_f(\gamma))(x):=f(x)\cdot\gamma(x)$$

is a continuous $\mathbb{K}$-linear map.

Proof. Arguing as before, we find that it suffices to discuss the case where $r \in \mathbb{N}_0$. The proof is by induction. In the following, let us write $m_{f,r}$ for m_f, to stress its dependence on r. Given $r \geq s \in \mathbb{N}_0$, we let $i_{s,r}\colon C^r(U,E) \to C^s(U,E)$ be the inclusion map.

Case $r = 0$: Let $K \subseteq U$ be a compact subset and $V \subseteq E$ be an open zero-neighbourhood. Since $f(K)$ is compact, $f(K) \cdot 0 \subseteq \{0\} \subseteq V$, and scalar multiplication is continuous, there is an open zero-neighbourhood $W \subseteq E$ such that $f(K) \cdot W \subseteq V$. As a consequence, $m_f\big(\lfloor K, W\rfloor\big) \subseteq \lfloor K, V\rfloor$. Being continuous at 0 by the preceding, the linear map $m_{f,0}$ is continuous.

Induction step. Suppose that the assertion of the lemma is correct for some $r \in \mathbb{N}_0$, and let $f\colon U \to \mathbb{K}$ be a mapping of class $C^{r+1}_{\mathbb{F}}$. Then $i_{0,r+1} \circ m_{f,r+1} = m_{f,0} \circ i_{0,r+1}$ shows that $i_{0,r+1} \circ m_{f,r+1}$ is a continuous linear map. Using that scalar multiplication $\beta\colon \mathbb{K} \times E \to E$ is a continuous $\mathbb{K}$-bilinear (and thus $\mathbb{F}$-bilinear) map, the formula for $\beta^{[1]}$ (see Examples 1.6) combined with the Chain Rule shows that

$$\begin{aligned}
&(m_{f,r+1}(\gamma))^{[1]}(x,y,t)\\
&\quad= (\beta \circ (f,\gamma))^{[1]}(x,y,t)\\
&\quad= \beta(f(x), \gamma^{[1]}(x,y,t)) + \beta(f^{[1]}(x,y,t), \gamma(x)) + t\beta(f^{[1]}(x,y,t), \gamma^{[1]}(x,y,t))\\
&\quad= f(x) \cdot \gamma^{[1]}(x,y,t) + f^{[1]}(x,y,t) \cdot \gamma(x) + tf^{[1]}(x,y,t) \cdot \gamma^{[1]}(x,y,t),
\end{aligned}$$

whence

$$(m_{f,r+1}(\gamma))^{[1]} = (m_{f\circ\pi + \tau\cdot f^{[1]},r} \circ \phi)(\gamma) + (m_{f^{[1]},r} \circ C^r(\pi, E) \circ i_{r,r+1})(\gamma)\,, \tag{9}$$

where $\pi\colon U^{[1]} \to U$, $\pi(x,y,t) := x$ and $\tau\colon U^{[1]} \to \mathbb{K}$, $\tau(x,y,t) := t$ are smooth and thus $C^r_{\mathbb{F}}$, multiplication operators are denoted in the apparent way, and $\phi\colon C^{r+1}(U,E) \to C^r(U^{[1]}, E)$ denotes the continuous linear map $\gamma \mapsto \gamma^{[1]}$. In view of the induction hypothesis and Lemma 4.4, Equation (9) shows that $C^{r+1}(U,E) \to C^r(U^{[1]},E)$, $\gamma \mapsto (m_{f,r+1}(\gamma))^{[1]}$ is a continuous $\mathbb{K}$-linear map. By Remark 4.2 (b), $m_{f,r+1}$ is continuous. □

Lemma 4.6 *Let Z be a topological $\mathbb{F}$-vector space, $U \subseteq Z$ be an open subset, and $(U_i)_{i\in I}$ be an open cover of U. For $i \in I$, let $\lambda_i\colon U_i \hookrightarrow U$ be the inclusion map, and*

$$\rho_i := C^r(\lambda_i, E)\colon C^r(U,E) \to C^r(U_i, E), \quad \rho_i(\gamma) := \gamma|_{U_i}$$

be the corresponding restriction map. Then the topology on $C^r(U,E)$ is initial with respect to the family $(\rho_i)_{i\in I}$.

Proof. Arguing as usual, we may assume that r is finite. The proof is by induction. By Lemma 4.4, each map ρ_i is continuous linear and thus the initial topology $\mathcal{O}_r$ on $C^r(U,E)$ with respect to $(\rho_i)_{i\in I}$ is a (Hausdorff) vector topology on $C^r(U,E)$ which is coarser than the given topology. We shall write $\rho_{i,r}$ for ρ_i, to stress its dependence on r.

The case $r = 0$. Suppose that K is a compact subset of U, and $W \subseteq E$ an open zero-neighbourhood. Given $x \in K$, there exists $i \in I$ such that $x \subseteq U_i$. Since K is compact Hausdorff, there exists a compact neighbourhood V_x of x in K such that $V_x \subseteq K \cap U_i$. As a consequence, using the compactness of K we find finitely many compact subsets $A_1, \ldots, A_n$ of K covering K, such that, for each $j = 1, \ldots, n$, there is $i_j \in I$ with $A_j \subseteq U_{i_j}$. Then $\lfloor K, W \rfloor \subseteq C^0(U, E)$ coincides with $\bigcap_{j=1}^n \rho_j^{-1}(\lfloor A_j, W \rfloor)$, where $\lfloor A_j, W \rfloor \subseteq C^0(U_j, E)$. As a consequence, the vector topology $\mathcal{O}_0$ on $C^0(U, E)$ is finer than the given topology and thus coincides with it.

Induction step. Suppose that the assertion of the lemma is correct for some $r \in \mathbb{N}_0$. In view of Remark 4.2 (b), we have to show that the mappings

$$\phi\colon (C^{r+1}(U, E), \mathcal{O}_{r+1}) \to C^0(U, E), \quad \phi(x) = x \quad \text{and}$$
$$\psi\colon (C^{r+1}(U, E), \mathcal{O}_{r+1}) \to C^r(U^{[1]}, E), \quad \psi(\gamma) := \gamma^{[1]}$$

are continuous, using the usual topology on the spaces on the right hand side. Let $j_i : C^{r+1}(U_i, E) \to C^0(U_i, E)$ be the inclusion map, which is continuous linear. As $\rho_{i,0} \circ \phi = j_i \circ \rho_{i,r+1}$ is continuous, we deduce from the C^0-case of the lemma already proved that ϕ is continuous.

To see that also ψ is continuous, let $(x, y, t) \in U^{]1[}$ and $\gamma \in C^{r+1}(U, E)$ (see **1.8**). Then

$$\psi(\gamma)(x, y, t) = \tfrac{1}{t}(\gamma(x + ty) - \gamma(x))$$

and thus

$$\psi(\gamma)|_{U^{]1[}} = (m_f \circ (C^r(s, E) - C^r(\pi, E)) \circ \mu_{r,r+1})(\gamma), \tag{10}$$

where the inclusion map $\mu_{r,r+1}\colon (C^{r+1}(U, E), \mathcal{O}_{r+1}) \to (C^r(U, E), \mathcal{O}_r) = C^r(U, E)$ (induction hypothesis!) is apparently continuous linear, and $f : U^{]1[} \to \mathbb{K}$, $(x, y, t) \mapsto t^{-1}$ is of class $C^r_{\mathbb{F}}$ as well as the mappings $s\colon U^{]1[} \to U$, $s(x, y, t) := x + ty$ and $\pi : U^{]1[} \to U$, $\pi(x, y, t) := x$. By Lemma 4.5, the multiplication operator $m_f\colon C^r(U^{]1[}, E) \to C^r(U^{]1[}, E)$ is continuous, and by Lemma 4.4, the mappings $C^r(s, E)$ and $C^r(\pi, E)$ are continuous. Thus Equation (10) shows that

$$(C^{r+1}(U, E), \mathcal{O}_{r+1}) \to C^r(U^{]1[}, E), \quad \gamma \mapsto \psi(\gamma)|_{U^{]1[}} \tag{11}$$

is a continuous mapping.

Next, suppose that $p = (x_0, y_0, t_0) \in U^{[1]}$ is given such that $t_0 = 0$. There exists $i \in I$ such that $x_0 \in U_i$. Then $(x_0, y_0, 0) \in (U_i)^{[1]}$, which is an open subset of $U^{[1]}$. As $\rho_{i,r+1}$ is continuous on $(C^{r+1}(U, E), \mathcal{O}_{r+1})$ and also $C^{r+1}(U_i, E) \to C^r(U_i^{[1]}, E)$, $\gamma \mapsto \gamma^{[1]}$ is continuous, we deduce that the mapping $(C^{r+1}(U, E), \mathcal{O}_{r+1}) \to C^r(U_i^{[1]}, E)$,

$$\gamma \mapsto (\gamma|_{U_i})^{[1]} = \gamma^{[1]}|_{U_i^{[1]}} = \psi(\gamma)|_{U_i^{[1]}} \tag{12}$$

is continuous. Now $\{U^{]1[}\} \cup \{U_i^{[1]} : i \in I\}$ being an open cover of $U^{[1]}$, using the induction hypothesis we deduce from the continuity of the mappings described in (11) and (12) that ψ is continuous. $\square$

The spaces $C^r(M,E)$ and mappings between them

4.7 Given a topological $\mathbb{K}$-vector space E and $\mathbb{F}$-manifold M of class $C^r_{\mathbb{F}}$, modeled on a topological $\mathbb{F}$-vector space Z, we let $C^r(M,E)$ be the set of all mappings $\gamma\colon M\to E$ of class $C^r_{\mathbb{F}}$. It is clear that pointwise operations turn $C^r(M,E)$ into a $\mathbb{K}$-vector space. We give $C^r(M,E)$ the initial topology with respect to the mappings

$$\theta_\kappa\colon C^r(M,E)\to C^r(V,E),\quad \gamma\mapsto \gamma|_U\circ\kappa^{-1}\,, \tag{13}$$

where $\kappa\colon U\to V\subseteq Z$ ranges through the charts of M. It is clear that this topology makes $C^r(M,E)$ a topological $\mathbb{K}$-vector space.

4.8 Since an open subset $U\subseteq Z$ may be considered as an $\mathbb{F}$-manifold, we now have two definitions of a topology on $C^r(U,E)$, described in **4.1** and **4.7**. As a consequence of Lemma 4.6, both topologies coincide:

Lemma 4.9 *If $\mathcal{A}$ is an atlas of charts for M, then the topology on $C^r(M,E)$ is initial with respect to the family $(\theta_\kappa)_{\kappa\in\mathcal{A}}$.*

Proof. Apparently, the initial topology $\mathcal{O}$ with respect to $(\theta_\kappa)_{\kappa\in\mathcal{A}}$ is coarser than the given topology on $C^r(M,E)$. To see that it is also finer, we have to show that $\mathcal{O}$ makes θ_η continuous, for every chart $\eta: U\to V$ of M. For $\kappa\in\mathcal{A}$, say $\kappa: U_\kappa\to W_\kappa$, define $V_\kappa:=\eta(U_\kappa\cap U)$. Then $(V_\kappa)_{\kappa\in\mathcal{A}}$ is an open cover of V, and as $(C^r(M,E),\mathcal{O})\to C^r(V_\kappa,E)$,

$$\gamma\mapsto\theta_\eta(\gamma)|_{V_\kappa}=\gamma\circ\eta^{-1}|_{V_\kappa}=\theta_\kappa(\gamma)\circ\kappa\circ\eta^{-1}|_{V_\kappa}=(C^r(\kappa\circ\eta^{-1}|_{V_\kappa},E)\circ\theta_\kappa)(\gamma)$$

is a continuous function of γ by Lemma 4.4 and definition of $\mathcal{O}$, for each $\kappa\in\mathcal{A}$, we deduce from Lemma 4.6 that θ_η is continuous, which completes the proof. □

Remark 4.10 The topology on $C^0(M,E)=C(M,E)$ just defined coincides with the compact-open topology. Indeed, the new topology obviously is coarser than the compact open topology, but it is also finer, by the argument used in the proof of Lemma 4.6, case $r=0$ (see also Lemma 4.22).

Lemma 4.11 *Let M and N be $C^r_{\mathbb{F}}$-manifolds modeled on topological $\mathbb{F}$-vector spaces, E be a topological $\mathbb{K}$-vector space, and $f\colon M\to N$ be a $C^r_{\mathbb{F}}$-map. Then the "pullback"*

$$C^r(f,E)\colon C^r(N,E)\to C^r(M,E),\quad \gamma\mapsto\gamma\circ f$$

is a continuous $\mathbb{K}$-linear map.

Proof. It is clear that $C^r(f,E)$ is $\mathbb{K}$-linear. There exists an atlas $\{\kappa_i\colon i\in I\}$ of charts $\kappa_i\colon U_i\to V_i$ of M such that, for each $i\in I$, $f(U_i)\subseteq A_i$ for some chart $\phi_i\colon A_i\to B_i$ of N. Given $i\in I$, consider $\theta_i\colon C^r(M,E)\to C^r(V_i,E)$, $\theta_i(\gamma):=\gamma\circ\kappa_i^{-1}$ and $\Theta_i\colon C^r(N,E)\to C^r(B_i,E)$, $\Theta_i(\gamma):=\gamma\circ\phi_i^{-1}$. In view of Lemma 4.9, the mapping $C^r(f,E)$ is continuous if and only if $\theta_i\circ C^r(f,E)$ is continuous for each i. But $\theta_i\circ C^r(f,E)=C^r(\phi_i\circ f|_{U_i}^{A_i}\circ\kappa_i^{-1},E)\circ\Theta_i$ is a composition of continuous mappings (see Lemma 4.4) □

Lemma 4.12 *Let M be a $C^r_{\mathbb{F}}$-manifold, modeled on a topological $\mathbb{F}$-vector space, E be a topological $\mathbb{K}$-vector space, and $(U_i)_{i\in I}$ be an open cover of M. Then*

$$\rho := (\rho_i)_{i\in I}\colon C^r(M,E) \to \prod_{i\in I} C^r(U_i,E), \quad \gamma \mapsto (\gamma|_{U_i})_{i\in I}$$

is a topological embedding, whose image is a closed vector subspace of $\prod_{i\in I} C^r(U_i,E)$.

Proof. Let $\lambda_i : U_i \to M$ be the inclusion maps. The coordinate functions of ρ are the restriction maps $\rho_i = C^r(\lambda_i,E)\colon C^r(M,E) \to C^r(U_i,E)$, $\gamma \mapsto \gamma\circ\lambda_i = \gamma|_{U_i}$, which are continuous linear by Lemma 4.11. Hence ρ is continuous linear, and apparently it is injective. Let $\mathcal{A}$ be the set of all charts of M whose domain is contained in some U_i. Then $\mathcal{A}$ is an atlas for M. Given $\kappa \in \mathcal{A}$, say $\kappa : U \to V$ with $U \subseteq U_i$, we can write $\theta_\kappa\colon C^r(M,E) \to C^r(V,E)$, $\gamma \mapsto \gamma\circ\kappa^{-1}$ as $\theta_\kappa = \Theta_\kappa \circ \rho_i$, where $\Theta_\kappa\colon C^r(U_i,E) \to C^r(V,E)$, $\eta \mapsto \eta\circ\kappa^{-1}$. As a consequence, θ_κ is continuous with respect to the topology $\mathcal{O}$ induced by ρ on $C^r(M,E)$. Hence, by Lemma 4.9, $\mathcal{O}$ has to be finer than the given topology on $C^r(M,E)$. Being also coarser (since ρ is continuous), it coincides with the given topology. Thus ρ is a topological embedding.

Let $F := \operatorname{im}\rho$, and $\overline{F}$ be its closure. Given $j,k \in I$, and $x \in U_j \cap U_k$, define $f_{j,k,x} : \prod_{i\in I} C^r(U_i,E) \to E$ via $(\gamma_i)_{i\in I} \mapsto \gamma_j(x) - \gamma_k(x)$. Then $f_{j,k,x}$ is a continuous linear map. From $f_{j,k,x}(F) = \{0\}$ we deduce $f_{j,k,x}(\overline{F}) \subseteq \overline{\{0\}} = \{0\}$. Thus $\gamma_j|_{U_j\cap U_k} = \gamma_k|_{U_j\cap U_k}$ for all $(\gamma_i)_{i\in I} \in \overline{F}$ and $j,k \in I$. As a consequence, given $(\gamma_i)_{i\in I} \in \overline{F}$, we can unambiguously define a mapping $\gamma\colon U \to E$ via $\gamma(x) := \gamma_i(x)$ if $x \in U_i$. Since $\gamma|_{U_i} = \gamma_i$ is of class C^r for each $i \in I$, Lemma 1.12 shows that γ is a mapping of class C^r. It remains to note that $(\gamma_i)_{i\in I} = \rho(\gamma) \in F$. □

Various simple observations will be useful.

Lemma 4.13 *Suppose that $\lambda\colon E \to F$ is a continuous $\mathbb{K}$-linear map between topological $\mathbb{K}$-vector spaces. Then*

$$C^r(M,\lambda)\colon C^r(M,E) \to C^r(M,F), \quad \gamma \mapsto \lambda\circ\gamma$$

is a continuous linear map.

Proof. Given a chart $\kappa : U \to V$ of M, we have $\theta^F_\kappa \circ C^r(M,\lambda) = C^r(V,\lambda)\circ\theta^E_\kappa$, where $\theta^E_\kappa : C^r(M,E) \to C^r(V,E)$, $\gamma \mapsto \gamma\circ\kappa^{-1}$, and $\theta^F_\kappa : C^r(M,F) \to C^r(V,F)$ is defined analogously. The topology on $C^r(M,F)$ being initial with respect to the mappings θ^F_κ, it therefore suffices to show that $C^r(V,\lambda)$ is continuous, for any open subset V of the modeling space of M. Let $j \in \mathbb{N}_0$ such that $j \le r$. For each $\gamma \in C^r(V,E)$, we have $(C^r(V,\lambda)(\gamma))^{[j]} = (\lambda\circ\gamma)^{[j]} = \lambda\circ(\gamma^{[j]}) = C(V^{[j]},\lambda)(\gamma^{[j]})$ since λ is continuous linear. Here $C^r(V,E) \to C(V^{[j]},E)$, $\gamma \mapsto \gamma^{[j]}$ is continuous linear by definition of the C^r-topology, and $C(V^{[j]},\lambda)\colon C(V^{[j]},E) \to C(V^{[j]},F)$, $\eta \mapsto \lambda\circ\eta$ is continuous with respect to the compact-open topologies by [16, §3.4, Assertion (1)]. The topology on $C^r(V,F)$ being initial with respect to the maps $(\bullet)^{[j]}$, we deduce that $C^r(V,\lambda)$ is continuous. □

If the topology on a topological space X is initial with respect to a family of maps into topological spaces whose topology is again initial with respect to certain families of maps,

then the topology on X is initial with respect to the family of composed maps. This well-known fact will be referred to as "transitivity of initial topologies" in the following.

Lemma 4.14 *Suppose that the topology on E is initial with respect to a family $(\lambda_i)_{i\in I}$ of $\mathbb{K}$-linear maps $\lambda_i : E \to E_i$ into topological $\mathbb{K}$-vector spaces E_i. Then the topology on $C^r(M,E)$ is initial with respect to the family $(C^r(M,\lambda_i))_{i\in I}$ of the linear mappings $C^r(M,\lambda_i)\colon C^r(M,E) \to C^r(M,E_i)$.*

Proof. The topologies on $C^r(M,E)$ and $C^r(M,E_i)$ are initial with respect to the mappings $\theta_\kappa : C^r(M,E) \to C^r(V_\kappa,E)$, resp., $\theta_{\kappa,i} : C^r(M,E_i) \to C^r(V_\kappa,E_i)$ (as in (13)), where $\kappa\colon U_\kappa \to V_\kappa$ ranges through the set of charts of M. Hence, we deduce from $C^r(V_\kappa,\lambda_i)\circ\theta_\kappa = \theta_{\kappa,i}\circ C^r(M,\lambda_i)$ and the transitivity of initial topologies that the assertion will hold if we can prove it when M is an open subset of a topological $\mathbb{F}$-vector space Z (like the sets V_κ). Using Remark 4.2 (a) in a similar way, we may furthermore assume that $r \in \mathbb{N}_0$ is finite. Now, the proof is by induction.

For $r = 0$, in view of Remark 4.10 the assertion is immediate from [16, La. 3.4.6]. If $r \in \mathbb{N}$ and the assertion holds when r is replaced with $r-1$, we recall that, for $M = U \subseteq Z$, the topology on $C^r(U,E)$ is initial with respect to the inclusion map $f\colon C^r(U,E) \to C(U,E)$ and the map $(\bullet)^{[1]}\colon C^r(U,E) \to C^{r-1}(U^{[1]},E)$. Let $f_i\colon C^r(U,E_i) \to C(U,E_i)$ be inclusion. Since $C^{r-1}(U^{[1]},\lambda_i)\circ(\bullet)^{[1]} = (\bullet)^{[1]}\circ C^r(U,\lambda_i)$ and $C(U,\lambda_i)\circ f = f_i \circ C^r(U,\lambda_i)$, we deduce from the induction hypothesis, the case $r=0$ and the transitivity of initial topologies that the topology on $C^r(M,E)$ is indeed initial with respect to the maps $C^r(U,\lambda_i)$. □

As an immediate consequence, we have:

Lemma 4.15 *Let E_1 and E_2 be topological $\mathbb{K}$-vector spaces, and $\mathrm{pr}_1 : E_1\times E_2 \to E_1$, $\mathrm{pr}_2\colon E_1\times E_2 \to E_2$ be the coordinate projections. Then*

$$\big(C^r(M,\mathrm{pr}_1),\, C^r(M,\mathrm{pr}_2)\big) : C^r(M,E_1\times E_2) \to C^r(M,E_1)\times C^r(M,E_2)$$

is an isomorphism of topological $\mathbb{K}$-vector spaces. □

Using the latter isomorphism, we shall frequently identify a function $\gamma \in C^r(M,E_1\times E_2)$ with its pair of coordinate functions (γ_1,γ_2), $\gamma_i := \mathrm{pr}_i\circ\gamma$.

Proposition 4.16 *Let E, F, H and $\widetilde{Z}$ be topological $\mathbb{K}$-vector spaces, $P \subseteq H$ be an open subset, $r,k \in \mathbb{N}_0\cup\{\infty\}$, $\widetilde{M}$ be a $\mathbb{K}$-manifold of class $C^{r+k}_{\mathbb{K}}$ modeled on $\widetilde{Z}$, and $\tilde{f}\colon \widetilde{M}\times E\times P \to F$ be a mapping of class $C^{r+k}_{\mathbb{K}}$. Let M be an $\mathbb{F}$-manifold of class $C^r_{\mathbb{F}}$, modeled on a topological $\mathbb{F}$-vector space Z. Given a $C^r_{\mathbb{F}}$-map $\sigma\colon M \to \widetilde{M}$, define*

$$f := \tilde{f}\circ(\sigma\times\mathrm{id}_E\times\mathrm{id}_P)\colon M\times E\times P \to F\,.$$

Then

$$\phi\colon C^r(M,E)\times P \to C^r(M,F),\quad \phi(\gamma,p) := f(\bullet,p)_*(\gamma)$$

is a mapping of class $C^k_{\mathbb{K}}$, where $f(\bullet,p)_*(\gamma)(x) := f(x,\gamma(x),p)$ for $x \in M$.

Proof. Since $C^\infty(M,F) = \lim\limits_{\longleftarrow\, \ell\in\mathbb{N}_0} C^\ell(M,F)$ apparently and the inclusion map $C^\infty(M,E)\to C^\ell(M,E)$ is continuous linear and thus of class $C^\infty_{\mathbb{K}}$, we easily deduce with Lemma 1.17 that ϕ is of class $C^\infty_{\mathbb{K}}$ in the case $r=\infty$, provided the proposition holds for all $r\in\mathbb{N}_0$. It therefore suffices to consider the case $r\in\mathbb{N}_0$.

Reduction to open subsets of Z and $\widetilde{Z}$

There is an atlas $\mathcal{A}=\{\kappa_i : i\in I\}$ of charts $\kappa_i : U_i\to V_i\subseteq Z$ of M such that $\sigma(U_i)$ is contained in the domain S_i of a chart $\tau_i : S_i\to R_i\subseteq \widetilde{Z}$ of $\widetilde{M}$. In view of Lemma 4.12, Lemma 4.11, Lemma 1.15 and Lemma 1.16, the map ϕ will be $C^k_{\mathbb{K}}$ if we can show that

$$h_i\colon C^r(M,E)\times P\to C^r(V_i,F),\quad (\gamma,p)\mapsto \phi(\gamma,p)\circ\kappa_i^{-1}$$

is of class $C^k_{\mathbb{K}}$, for every $i\in I$. Then $\tilde{f}_i:=\tilde{f}\circ(\tau_i^{-1}\times\mathrm{id}_E\times\mathrm{id}_P)\colon R_i\times E\times P\to F$ is a $C^{r+k}_{\mathbb{K}}$-map, and $\sigma_i:=\tau_i\circ\sigma|^{S_i}_{U_i}\circ\kappa_i^{-1}\colon V_i\to R_i$ is of class $C^r_{\mathbb{F}}$. We set $f_i:=\tilde{f}_i\circ(\sigma_i\times\mathrm{id}_E\times\mathrm{id}_P)\colon V_i\times E\times P\to F$ and define

$$\phi_i\colon C^r(V_i,E)\times P\to C^r(V_i,F),\quad \phi_i(\gamma,p):=f(\bullet,p)\circ(\mathrm{id}_{V_i},\gamma)\,.$$

In view of Lemma 4.11, the formula $h_i(\gamma,p)=\phi_i(C^r(\kappa_i^{-1},E)(\gamma),p)$ shows that h_i will be $C^k_{\mathbb{K}}$ if so is ϕ_i. Replacing M with V_i and $\widetilde{M}$ with R_i, we may therefore assume that M and $\widetilde{M}$ are open subsets of Z, resp., $\widetilde{Z}$, for the rest of the proof.

Apparently, it suffices to consider the case where $k\in\mathbb{N}_0$; the proof is by induction on k.

The case $k=0$.

The proof is by induction on r. If $r=0$, then the topology on $C^0(M,E)$ and $C^0(M,F)$ is the topology of uniform convergence on compact sets (see **4.8**). Let $\gamma\in C(M,E)$, $p\in P$, L be a compact subset of M, and $V\subseteq F$ be an open zero-neighbourhood. Let $W\subseteq F$ be an open zero-neighbourhood such that $W-W\subseteq V$. For each $x\in L$, we find an open neighbourhood $A_x\subseteq M$ of x and open zero-neighbourhoods $B_x\subseteq E$ and $C_x\subseteq H$ such that $p+C_x\subseteq P$ and

$$f(y,u,q)-f(x,\gamma(x),p)\in W$$

for all $y\in A_x$, $u\in\gamma(A_x)+B_x$, and $q\in p+C_x$. By compactness, $L\subseteq\bigcup_{x\in I}A_x$ for some finite subset $I\subseteq L$. Then $B:=\bigcap_{x\in I}B_x\subseteq E$ and $C:=\bigcap_{x\in I}C_x\subseteq H$ are open zero-neighbourhoods. Let $\xi\in\gamma+\lfloor L,B\rfloor$ and $q\in p+C\subseteq P$. Given $y\in L$, there is $x\in I$ such that $y\in A_x$. Then

$$\begin{aligned} f(y,\xi(y),q)-f(y,\gamma(y),p) &= f(y,\xi(y),q)-f(x,\gamma(x),p)-(f(y,\gamma(y),p)-f(x,\gamma(x),p))\\ &\in W-W\subseteq V. \end{aligned}$$

We have shown that $\phi(\xi,q)-\phi(\gamma,p)\in\lfloor L,V\rfloor\subseteq C(M,F)$ for all (ξ,q) in the open neighbourhood $(\gamma+\lfloor L,B\rfloor)\times(p+C)$ of (γ,p). Thus ϕ is continuous, as required.

Induction step on r. We write ϕ_r for ϕ, to emphasize its dependence on r. Suppose the assertion of the lemma is correct for $k=0$ and some $r\in\mathbb{N}_0$. Suppose that the hypotheses of the lemma are satisfied by $\tilde{f}$ and σ, with r replaced by $r+1$. Let $i\colon C^{r+1}(M,E)\to C(M,E)$ and $j\colon C^{r+1}(M,F)\to C(M,F)$ be the inclusion maps. The mapping

$$C^{r+1}(M,F)\to C(M,F)\times C^r(M^{[1]},F),\quad \gamma\mapsto(\gamma,\gamma^{[1]})$$

is an embedding of topological $\mathbb{K}$-vector spaces by Remark 4.2 (b). Thus ϕ_{r+1} will be continuous if we can show that the mappings $j\circ\phi_{r+1}$ and

$$\psi\colon C^{r+1}(M,E)\times P\to C^r(M^{[1]},F),\quad \psi(\gamma,p):=\phi_{r+1}(\gamma,p)^{[1]}$$

are continuous. We already know that ϕ_0 is continuous, whence $j\circ\phi_{r+1}=\phi_0\circ(i\times\mathrm{id}_P)$ is continuous. Recall that $\phi_{r+1}(\gamma,p)(x)=\tilde{f}(\sigma(x),\gamma(x),p)$ for $\gamma\in C^{r+1}(M,E)$, $p\in P$ and $x\in M$. The Chain Rule gives

$$\begin{aligned}\psi(\gamma,p)(x,y,t) &= \phi_{r+1}(\gamma,p)^{[1]}(x,y,t)\\ &= \tilde{f}^{[1]}((\sigma(x),\gamma(x),p),\ (\sigma^{[1]}(x,y,t),\gamma^{[1]}(x,y,t),0),\ t)\end{aligned}$$

for all $\gamma\in C^{r+1}(M,E)$, $p\in P$ and $(x,y,t)\in M^{[1]}$. Hence

$$\psi(\gamma,p)=g(\bullet,p)_*(\gamma\circ\mathrm{pr}_1,\gamma^{[1]})\,, \tag{14}$$

where $\mathrm{pr}_1\colon M^{[1]}\to M$, $(x,y,t)\mapsto x$, and $g:=\tilde{g}\circ\Big((\widehat{T}\sigma)\times\mathrm{id}_{E^2}\times\mathrm{id}_P\Big)\colon M^{[1]}\times E^2\times P\to F$ with

$$\tilde{g}\colon\widetilde{M}^{[1]}\times E^2\times P\to F,\quad \tilde{g}((x,y,t),\,(u,v),\,p):=\tilde{f}^{[1]}((x,u,p),\,(y,v,0),\,t)$$

of class $C^r_{\mathbb{K}}$ and $\widehat{T}\sigma\colon M^{[1]}\to\widetilde{M}^{[1]}$, $(\widehat{T}\sigma)(x,y,t):=(\sigma(x),\sigma^{[1]}(x,y,t),t)$ of class $C^r_{\mathbb{F}}$. By the induction hypothesis, the map

$$C^r(M^{[1]},E^2)\times P\to C^r(M^{[1]},F),\quad (\kappa,p)\mapsto g(\bullet,p)_*(\kappa)$$

is continuous. As $C^{r+1}(M,E)\to C^r(M^{[1]},E^2)\cong C^r(M^{[1]},E)^2$, $\gamma\mapsto(\gamma\circ\mathrm{pr}_1,\gamma^{[1]})$ is continuous as well (cf. Remark 4.2, Lemma 4.4, Lemma 4.15), we deduce from (14) that ψ is continuous, and hence so is ϕ_{r+1}.

Induction step on k.

Suppose the assertion of the lemma is correct for some $k\in\mathbb{N}_0$ and all $r\in\mathbb{N}_0$. Let σ and $\tilde{f}$ be given which satisfy the hypotheses of the lemma when k is replaced with $k+1$. Then $\phi\colon C^r(M,E)\times P\to C^r(M,F)$ is of class $C^k_{\mathbb{K}}$ (and thus continuous), by induction. Given $\gamma,\eta\in C^r(M,E)$, $p,q\in H$ and $t\in\mathbb{F}$, we clearly have $(\gamma,p,\eta,q,t)\in(C^r(M,E)\times P)^{[1]}$ if and only if $(p,q,t)\in P^{[1]}$. In this case, provided $t\neq 0$ we calculate

$$\begin{aligned}&\tfrac{1}{t}(\phi(\gamma+t\eta,p+tq)-\phi(\gamma,p))\,(x)\\ &\quad= \tfrac{1}{t}(\tilde{f}(\sigma(x),\gamma(x)+t\eta(x),p+tq)-\tilde{f}(\sigma(x),\gamma(x),p))\\ &\quad= \tilde{f}^{[1]}((\sigma(x),\gamma(x),p),\ (0,\eta(x),q),\ t)\end{aligned}$$

for all $x \in M$. Hence

$$\frac{1}{t}(\phi(\gamma + t\eta, p + tq) - \phi(\gamma, p)) = h(\bullet, (p, q, t))_*(\gamma, \eta) \tag{15}$$

for all $\gamma, \eta \in C^r(M, E)$ and $(p, q, t) \in P^{[1]}$ such that $t \neq 0$, where $h := \tilde{h} \circ (\sigma \times \mathrm{id}_{E^2} \times \mathrm{id}_{P^{[1]}})$: $M \times E^2 \times P^{[1]} \to F$ arises from the $C^{r+k}_{\mathbb{K}}$-map

$$\tilde{h} \colon \widetilde{M} \times E^2 \times P^{[1]} \to F, \quad \tilde{h}(z, (u, v), (p, q, t)) := \tilde{f}^{[1]}((z, u, p), (0, v, q), t)\,.$$

By the induction hypothesis, the map

$$\psi \colon C^r(M, E^2) \times P^{[1]} \to C^r(M, F), \quad (\kappa, (p, q, t)) \mapsto h(\bullet, (p, q, t))_*(\kappa)$$

is of class $C^k_{\mathbb{K}}$ (and hence continuous). In view of (15), we see that ϕ is of class $C^1_{\mathbb{K}}$, with $\phi^{[1]}$ given by $\phi^{[1]}((\gamma, p), (\eta, q), t) = \psi((\gamma, \eta), (p, q, t))$ and thus of class $C^k_{\mathbb{K}}$. Hence ϕ is of class $C^{k+1}_{\mathbb{K}}$, as required. □

It would not make sense to omit the set of parameters P in the formulation of Proposition 4.16 (hoping to make the proof easier this way). In fact, even if P is a singleton, a non-singleton set P_1 will occur in the induction step on k of the preceding proof.

The spaces $C^r_K(M, E)$ and mappings between them

If $\mathbb{F}$ is locally compact and $K \subseteq M$ is a compact subset, we give

$$C^r_K(M, E) := \{\gamma \in C^r(M, E) \colon \mathrm{supp}(\gamma) \subseteq K\}$$

the topology induced by $C^r(M, E)$. The point evaluations $C^r(M, E) \to E$, $\gamma \mapsto \gamma(x)$ at the elements $x \in M$ being continuous linear maps, $C^r_K(M, E)$ is a closed vector subspace of $C^r(M, E)$. In the next proposition, we compile some useful properties of function spaces. The simple proofs are given in Appendix A. Only part (c) is needed for the Lie group constructions. Part (d) serves to put our studies in perspective. Before we can state the proposition, let us recall various concepts.

4.17 First, recall that a Hausdorff topological space X is called a *k-space* if a subset $U \subseteq X$ is open precisely if $U \cap K$ is open in K for every compact subset $K \subseteq X$. For example, every metrizable topological space is a k-space.

Definition 4.18 Let E be a topological $\mathbb{K}$-vector space.

(a) E is called *sequentially complete* if every Cauchy sequence in E is convergent.

(b) E is called *Mackey complete* if every Mackey-Cauchy sequence in E is convergent. Here, a sequence $(x_n)_{n \in \mathbb{N}}$ in E is called a *Mackey-Cauchy sequence*[6] if there exists a bounded subset $B \subseteq E$ and elements $\mu_{n,m} \in \mathbb{K}$ such that $x_n - x_m \in \mu_{n,m} B$ for all $n, m \in \mathbb{N}$ and $\mu_{n,m} \to 0$ in $\mathbb{K}$ as both $n, m \to \infty$.

[6] The two concepts mainly are of interest if $\mathbb{K}$ is a valued field.

Note that every Mackey-Cauchy sequence also is a Cauchy sequence; hence every sequentially complete topological $\mathbb{K}$-vector space is Mackey complete.

Proposition 4.19 *Let M be a $C^r_{\mathbb{F}}$-manifold, modeled on a topological $\mathbb{F}$-vector space Z. Let E be a topological $\mathbb{K}$-vector space. Then the following holds:*

(a) *Assume that $Z^{[j]}$ is a k-space for all $j \in \mathbb{N}_0$ such that $j \leq r$; for example, this holds if both $\mathbb{F}$ and Z are metrizable. Then $C^r(M,E)$ is complete (resp., sequentially complete, resp., Mackey complete) if E is complete (resp., sequentially complete, resp., Mackey complete).*

(b) *If $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$ or an ultrametric field and E is locally convex, then also $C^r(M,E)$ is locally convex.*

(c) *If $\mathbb{F}$ is locally compact, E is metrizable and M is a σ-compact, finite-dimensional $C^r_{\mathbb{F}}$-manifold, then $C^r(M,E)$ is metrizable.*

(d) *If $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{F},\mathbb{C}\}$ and E is locally convex, then the topology on $C^r(M,E)$ is initial with respect to the family $(D^j)_{r\geq j\in\mathbb{N}_0}$ of maps $D^j\colon C^r(M,E) \to C(T^jM,E)_{c.o.}$, $\gamma \mapsto D^j\gamma$, and hence it is the topology traditionally considered on $C^r(M,E)$ (see Appendix* A *for the notations).*

If $\mathbb{F}$ is locally compact, then analogous conclusions hold for the closed vector subspace $C^r_K(M,E)$ of $C^r(M,E)$, for every compact subset $K \subseteq M$. □

The remainder of this section is devoted to the following result (and variants), which will be needed, for example, for the discussion of groups of C^r-maps. Until the end of the section, we now assume that the topological field $\mathbb{F}$ is locally compact.

Proposition 4.20 *Let E, F, and $\widetilde{Z}$ be topological $\mathbb{K}$-vector spaces, $U \subseteq E$ an open subset, $r,k \in \mathbb{N}_0 \cup \{\infty\}$, $\widetilde{M}$ be a $\mathbb{K}$-manifold of class $C^{r+k}_{\mathbb{K}}$ modeled on $\widetilde{Z}$, and $\tilde{f}: \widetilde{M} \times U \to F$ be a mapping of class $C^{r+k}_{\mathbb{K}}$. Let M be a finite-dimensional $\mathbb{F}$-manifold of class $C^r_{\mathbb{F}}$, and $K \subseteq M$ be a compact subset. Given a mapping $\sigma: M \to \widetilde{M}$ of class $C^r_{\mathbb{F}}$, we define $f := \tilde{f} \circ (\sigma \times \mathrm{id}_U)\colon M \times U \to F$. If $K \neq M$, we assume that $0 \in U$ and $f(x,0) = 0$ for all $x \in M \setminus K$. Then $C^r_K(M,U) := \{\gamma \in C^r_K(M,E)\colon \gamma(M) \subseteq U\}$ is an open subset of $C^r_K(M,E)$, and*

$$f_*\colon C^r_K(M,U) \to C^r_K(M,F), \quad f_*(\gamma)(x) := f(x,\gamma(x))$$

is a mapping of class $C^k_{\mathbb{K}}$.

Corollary 4.21 *Let E and F be topological $\mathbb{K}$-vector spaces and $f\colon U \to F$ be a mapping of class $C^{r+k}_{\mathbb{K}}$, defined on an open subset U of E. Let M be a finite-dimensional $\mathbb{F}$-manifold of class $C^r_{\mathbb{F}}$, and $K \subseteq M$ a compact subset. If $K \neq M$, we suppose $0 \in U$ and $f(0) = 0$. Then*

$$C^r_K(M,f)\colon C^r_K(M,U) \to C^r_K(M,F), \quad \gamma \mapsto f \circ \gamma$$

is a mapping of class $C^k_{\mathbb{K}}$.

Proof. Let $\widetilde{M} := \{0\}$ be a singleton smooth $\mathbb{K}$-manifold, and $\sigma\colon M \to \widetilde{M}$, $x \mapsto 0$, which apparently is a smooth mapping. Then $\tilde{g}\colon \widetilde{M} \times U \to F$, $\tilde{g}(0,u) := f(u)$ is a mapping of class $C^{r+k}_{\mathbb{K}}$, and $C_K(M,f) = g_*$ for $g := \tilde{g} \circ (\sigma \times \mathrm{id}_U)$. By Proposition 4.20, g_* is $C^k_{\mathbb{K}}$. □

Instead of proving Proposition 4.20 directly, we deduce it from a more flexible technical result (Proposition 4.23 below), which shall be re-used repeatedly afterwards. For its formulation, we require certain sets $\lfloor K, U \rfloor_r$:

Lemma 4.22 *For every compact subset $K \subseteq M$ and open subset $U \subseteq E$, the set*

$$\lfloor K, U \rfloor_r := \{\gamma \in C^r(M,E) \colon \gamma(K) \subseteq U\}$$

is open in $C^r(M,E)$.

Proof. There are compact subsets $A_1, \ldots, A_n \subseteq K$ which cover K, and such that $A_i \subseteq U_i$ for some chart $\kappa_i \colon U_i \to V_i$ of M (cf. proof Lemma 4.6). Set $K_i := \kappa_i(A_i)$. Then $\lfloor K_i, U \rfloor \subseteq C(V_i, E)$ is open by definition of the compact-open topology, and $h_i \colon C^r(M,E) \to C(V_i,E)$, $h_i(\gamma) := \gamma \circ \kappa_i^{-1}$ is continuous, for each $i \in \{1,\ldots,n\}$. Thus $\lfloor K, U \rfloor_r = \bigcap_{i=1}^n \lfloor A_i, U \rfloor_r = \bigcap_{i=1}^n h_i^{-1}(\lfloor K_i, U \rfloor)$ is open in $C^r(M,E)$. □

We now formulate the main technical result of this section. For the moment, only part (a) of the proposition is needed, but the more general part (b) will become essential in the proof of Proposition 12.2 below (to tackle also the case of infinite-dimensional M there).

Proposition 4.23 *Let E, F, H and Z be topological $\mathbb{K}$-vector spaces, $U \subseteq E$ and $P \subseteq H$ be open subsets, $r, k \in \mathbb{N}_0 \cup \{\infty\}$, and N be a $\mathbb{K}$-manifold of class $C^{r+k}_{\mathbb{K}}$ modeled on Z. Let M be an $\mathbb{F}$-manifold of class $C^r_{\mathbb{F}}$, modeled on a finite-dimensional topological $\mathbb{F}$-vector space X, $K \subseteq M$ be a compact subset, $Y \subseteq K$ be an open, non-empty subset of M, and $\sigma\colon Y \to N$ be a mapping of class $C^r_{\mathbb{F}}$. Define $\lfloor K, U \rfloor_r \subseteq C^r(M,E)$ as above.*

(a) *If $\tilde{g}\colon N \times U \times P \to F$ is a $C^{r+k}_{\mathbb{K}}$-map and $g := \tilde{g} \circ (\sigma \times \mathrm{id}_U \times \mathrm{id}_P) \colon Y \times U \times P \to F$, then*

$$\lfloor K, U \rfloor_r \times P \to C^r(Y,F), \quad (\gamma, p) \mapsto g(\bullet, p)_*(\gamma)$$

is a mapping of class $C^k_{\mathbb{K}}$, where $g(\bullet,p)_*(\gamma)(x) := g(x, \gamma(x), p)$ for $x \in Y$.

(b) *More generally, let $\overline{E}$ be a topological $\mathbb{K}$-vector space, $\overline{M}$ an $\mathbb{F}$-manifold of class $C^r_{\mathbb{F}}$, modeled on a topological $\mathbb{F}$-vector space $\overline{X}$, and $\tilde{f}\colon N \times U \times \overline{E} \times P \to F$ be a $C^{r+k}_{\mathbb{K}}$-map. Define $f := \tilde{f} \circ (\sigma \times \mathrm{id}_U \times \mathrm{id}_{\overline{E}} \times \mathrm{id}_P) \colon Y \times U \times \overline{E} \times P \to F$. Then the map*

$$\phi\colon \lfloor K, U \rfloor_r \times C^r(\overline{M}, \overline{E}) \times P \to C^r(Y \times \overline{M}, F), \quad \phi(\gamma, \bar{\gamma}, p) := f(\bullet, p)_*(\gamma \times \bar{\gamma})$$

is of class $C^k_{\mathbb{K}}$, where $f(\bullet,p)_*(\gamma \times \bar{\gamma})(x, \bar{x}) := f(x, \gamma(x), \bar{\gamma}(\bar{x}), p)$ for $x \in Y$, $\bar{x} \in \overline{M}$.

Proof. The proof is similar to the one of Proposition 4.16, but longer and painfully technical in detail. We defer it to Appendix B. □

The following lemma helps to deduce Proposition 4.20 from Proposition 4.23:

Lemma 4.24 *If $K \subseteq M$ is a compact subset and $Y \subseteq M$ an open subset containing K, then the restriction map*

$$C^r_K(M,E) \to C^r_K(Y,E), \quad \gamma \mapsto \gamma|_Y$$

is an isomorphism of topological $\mathbb{K}$-vector spaces.

Proof. The restriction map clearly is an isomorphism of $\mathbb{K}$-vector spaces. Let $\mathcal{A}_1$ be an atlas for Y, and $\mathcal{A}_2$ an atlas for $M \setminus K$. Then $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$ is an atlas for M. For each $\kappa \in \mathcal{A}_2$, we have $\theta_\kappa(\gamma) = 0$ for each $\gamma \in C^r_K(M,E)$, entailing that the initial topology on $C^r_K(M,E)$ with respect to the mappings $\theta_\kappa|_{C^r_K(M,E)}$, where $\kappa \in \mathcal{A}$, coincides with the initial topology with respect to the subset of mappings parametrized by $\kappa \in \mathcal{A}_1$. The assertion now readily follows with Lemma 4.9. $\square$

Proof of Proposition 4.20. Let Z be the modeling space of M. Since $\mathbb{F}$ is locally compact, the canonical Hausdorff vector topology on the finite-dimensional $\mathbb{F}$-vector space Z is locally compact. Hence M is a locally compact topological space. We therefore find a relatively compact open neighbourhood Y of K in M. The inclusion mappings $i : C^r_K(M,E) \to C^r(M,E)$ and $j : C^r_K(Y,F) \to C^r(Y,F)$ are $\mathbb{K}$-linear and topological embeddings, with closed image. The restriction map $\rho\colon C^r_K(M,F) \to C^r_K(Y,F)$ is an isomorphism of topological $\mathbb{K}$-vector spaces by Lemma 4.24. Let $P := H := \{0\}$ (zero-dimensional $\mathbb{K}$-vector space), and define $g\colon Y \times U \times P \to F$, $g(x,y,p) := f(x,y)$. Then, by Proposition 4.23 (a), the map

$$\psi\colon \lfloor \overline{Y}, U \rfloor_r \times P \to C^r(Y,F), \quad \psi(\gamma,p) := g({\scriptstyle\bullet},p)_*(\gamma)$$

is of class $C^k_{\mathbb{K}}$, where $\lfloor \overline{Y}, U \rfloor_r \subseteq C^r(M,E)$. Note that $i(C^r_K(M,U)) = \lfloor \overline{Y}, U \rfloor_r \cap C^r_K(M,E)$. Thus $C^r_K(M,U)$ is open in $C^r_K(M,E)$, and $i(C^r_K(M,U)) \subseteq \lfloor \overline{Y}, U \rfloor_r$. Since $j \circ \rho \circ f_* = \psi({\scriptstyle\bullet},0) \circ i|^{\lfloor \overline{Y}, U \rfloor_r}_{C^r_K(M,U)}$ apparently, we see that $j \circ \rho \circ f_*$ is of class $C^k_{\mathbb{K}}$, whence so is f_*, by Lemma 1.15. $\square$

5 Mapping groups and mapping algebras

In this section, we discuss mapping groups and mapping algebras, based on our studies in Sections 3 and 4.

Throughout this section, $r \in \mathbb{N}_0 \cup \{\infty\}$. If $r = 0$, we let M be any topological space, and $\mathbb{K}$ any topological field. If $r > 0$, we let $\mathbb{F}$ be a locally compact topological field, M be a finite-dimensional $\mathbb{F}$-manifold of class $C^r_{\mathbb{F}}$, and $\mathbb{K}$ be a topological field possessing $\mathbb{F}$ as a topological subfield. In either case, we let $K \subseteq M$ be a compact subset.

Mapping Groups

Given a $\mathbb{K}$-Lie group G, we consider the set

$$C^r_K(M,G) := \{\gamma \in C^r(M,G) \colon \gamma|_{M\setminus K} = 1\}$$

of G-valued mappings of class $C^r_{\mathbb{F}}$ on M which are identically 1 off K.[7] It is clear that $C^r_K(M,G)$ is a group under pointwise multiplication and inversion. Then $C^r_K(M,G)$ is a $\mathbb{K}$-Lie group in a natural way:

Proposition 5.1 *On the group $C^r_K(M,G)$, there is a uniquely determined smooth $\mathbb{K}$-manifold structure with the following properties:*

(a) *it makes $C^r_K(M,G)$ a $\mathbb{K}$-Lie group; and:*

(b) *There exists a chart $\kappa \colon P \to Q$ from an open identity neighbourhood $P \subseteq G$ onto an open zero-neighbourhood $Q \subseteq L(G)$ such that $\kappa(1) = 0$, $T_1(\kappa) = \mathrm{id}_{L(G)}$, and such that $C^r_K(M,P) := C^r_K(M,G) \cap P^M$ is open in $C^r_K(M,G)$ and*

$$C^r_K(M,\kappa) \colon C^r_K(M,P) \to C^r_K(M,Q) \subseteq C^r_K(M,L(G)), \quad \gamma \mapsto \kappa \circ \gamma$$

is a diffeomorphism of smooth $\mathbb{K}$-manifolds.

Identifying $L(C^r_K(M,G))$ with $T_0(C^r_K(M,L(G))) = C^r_K(M,L(G))$ via $T_1(C^r_K(M,\kappa))$, the Lie bracket on $L(C^r_K(M,G))$ corresponds to the mapping $C^r_K(M,\beta) : C^r_K(M,L(G)^2) \cong C^r_K(M,L(G))^2 \to C^r_K(M,L(G))$, where $\beta \colon L(G)^2 \to L(G)$ is the Lie bracket of $L(G)$ (in other words, $[\gamma,\eta](x) = [\gamma(x),\eta(x)]$).

Proof. The following proof closely follows the lines of [19], Section 3, where only real and complex Lie groups modeled on locally convex spaces are considered. We proceed in steps.

5.2 Let $\phi \colon U_1 \to U$ be a chart of G, defined on an open identity neighbourhood U_1 in G, with values in an open zero-neighbourhood U in $L(G)$, such that $\phi(1) = 0$. Let V_1 be an open, symmetric identity neighbourhood in G such that $V_1V_1 \subseteq U_1$, and set $V := \phi(V_1)$. Then the mappings

$$\mu \colon V \times V \to U, \quad \mu(x,y) := \phi(\phi^{-1}(x)\cdot\phi^{-1}(y))$$

and

$$\iota \colon V \to V, \qquad \iota(x) := \phi(\phi^{-1}(x)^{-1})$$

are smooth. We equip $C^r_K(M,U_1) := \{\gamma \in C^r_K(M,G) : \gamma(M) \subseteq U_1\}$ with the smooth $\mathbb{K}$-manifold structure making the bijection

$$C^r_K(M,\phi) \colon C^r_K(M,U_1) \to C^r_K(M,U), \quad \gamma \mapsto \phi \circ \gamma$$

a diffeomorphism of smooth $\mathbb{K}$-manifolds onto the open subset $C^r_K(M,U) \subseteq C^r_K(M,L(G))$.

[7] To harmonize notation, we write $C^0(M,G) := C(M,G)$ now also in the case where M merely is a topological space, and call continuous mappings C^0-maps.

5.3 Since $C^r_K(M,V) \times C^r_K(M,V) \cong C^r_K(M, V \times V)$ as $C^\infty_{\mathbb{K}}$-manifolds (cf. Lemma 4.15), and $C^r_K(M,\mu) : C^r_K(M, V \times V) \to C^r_K(M,U)$ is $C^\infty_{\mathbb{K}}$ by Corollary 3.5 (resp., Corollary 4.21), we deduce that the group multiplication of $C^r_K(M,G)$ induces a $C^\infty_{\mathbb{K}}$-mapping $C^r_K(M,V_1) \times C^r_K(M,V_1) \to C^r_K(M,U_1)$. Similarly, inversion is $C^\infty_{\mathbb{K}}$ on $C^r_K(M,V_1)$.

5.4 Let $\gamma \in C^r_K(M,G)$ now. As $\gamma(M) \subseteq \gamma(K) \cup \{1\}$ is compact, there is an open identity neighbourhood $W_1 \subseteq V_1$ in G and an open neighbourhood P of $\gamma(M)$ in G such that $pW_1p^{-1} \subseteq U_1$ for all $p \in P$. Set $W := \phi(W_1)$. As $C^r_K(M,W)$ is open in $C^r_K(M,V)$, we deduce that $C^r_K(M,W_1)$ is open in $C^r_K(M,V_1)$. The mapping $h : P \times W_1 \to U_1$, $h(p,w) := pwp^{-1}$ being $C^\infty_{\mathbb{K}}$, also $\tilde{f} := \phi \circ h \circ (\mathrm{id}_P \times \phi^{-1}|^{W_1}_W) : P \times W \to U$ is $C^\infty_{\mathbb{K}}$. Then clearly the mapping $f := \tilde{f} \circ (\gamma \times \mathrm{id}_W) : M \times W \to U$, $f(x,y) = \phi(\gamma(x)\phi^{-1}(y)\gamma(x)^{-1})$ satisfies the hypotheses of Corollary 3.4 (resp., Proposition 4.20), with $k := \infty$. We deduce from Corollary 3.4 (resp., Proposition 4.20) that the mapping $f_* : C^r_K(M,W) \to C^r_K(M,U)$ is $C^\infty_{\mathbb{K}}$. Note that

$$C^r_K(M,\phi)^{-1} \circ f_* \circ C^r_K(M,\phi)\big|^{C^r_K(M,W)}_{C^r_K(M,W_1)} \;=\; I_\gamma\big|_{C^r_K(M,W_1)},$$

where $I_\gamma : C^r_K(M,G) \to C^r_K(M,G)$, $I_\gamma(\eta) := \gamma\eta\gamma^{-1}$. Thus $I_\gamma(C^r_K(M,W_1)) \subseteq C^r_K(M,U_1)$ and $I_\gamma\big|^{C^r_K(M,U_1)}_{C^r_K(M,W_1)}$ is $C^\infty_{\mathbb{K}}$ on the open identity neighbourhood $C^r_K(M,W_1) \subseteq C^r_K(M,V_1)$. Now Proposition 1.18 provides a unique smooth $\mathbb{K}$-manifold structure on $C^r_K(M,G)$ such that $C^r_K(M,G)$ becomes a $\mathbb{K}$-Lie group which possesses $C^r_K(M,V_1)$ as an open submanifold.

5.5 The Lie group $C^r_K(M,G)$ being modeled on $C^r_K(M,L(G))$, its Lie algebra can be identified with $C^r_K(M,L(G))$ as a topological vector space, by means of $T_1(C^r_K(M,\phi|^V_{V_1}))$. Let us show that the Lie bracket is the mapping $C^r_K(M,[.,.])$ on $C^r_K(M,L(G)^2) \cong C^r_K(M,L(G))^2$ (which is continuous by Corollary 3.5, resp., Corollary 4.21). To this end, note first that the point evaluation $\pi_x : C^r_K(M,G) \to G$, $\pi_x(\gamma) := \gamma(x)$ is a smooth homomorphism for each $x \in M$, since $\pi_x \circ C^r_K(M,\phi^{-1}|_V) = \phi^{-1}|_V \circ \Pi_x|^V_{C^r_K(M,V)}$ is smooth, using that the point evaluation $\Pi_x : C^r_K(M,L(G)) \to L(G)$ is a continuous linear map. As we identify $T_1C^r_K(M,G)$ with $C^r_K(M,L(G))$ by means of $T_1C^r_K(M,\phi|^V_{V_1})$, and $T_1\phi = \mathrm{id}_{L(G)}$ by hypothesis, we clearly have $L(\pi_x) = T_1(\pi_x) = \Pi_x$. As $L(\pi_x)$ is a Lie algebra homomorphism, we deduce that $[\gamma,\eta](x) = [\gamma(x),\eta(x)]$ for all $\gamma,\eta \in C^r_K(M,L(G))$. The assertion follows.

5.6 The asserted uniqueness of the Lie group structure on $C^r_K(M,G)$ with the required properties follows by standard arguments, using that $C^r_K(M,\kappa_1 \circ \kappa_2^{-1})$ is a diffeomorphism (by Corollary 3.5, resp., Corollary 4.21) if both κ_1 and κ_2 are charts of G with the described properties (whose domains coincide, without loss of generality). This completes the proof of Proposition 5.1. □

Mapping Algebras

Given an associative topological $\mathbb{K}$-algebra A (possibly without an identity element), we let A_e be the associated unital $\mathbb{K}$-algebra. Thus $A_e = A \oplus \mathbb{K}e$ as a $\mathbb{K}$-vector space. We give A_e the product topology, which makes it a unital, associative topological $\mathbb{K}$-algebra.

Proposition 5.7 *If A is a continuous inverse algebra over $\mathbb{K}$, then also $C^r_K(M,A)_e$ is a continuous inverse $\mathbb{K}$-algebra. In the special case where $M = K$ is compact, also $C^r(K,A)$ is a continuous inverse $\mathbb{K}$-algebra.*

Proof. Since we have Corollaries 3.5 and 4.21 at our disposal, the arguments used in [20] to prove the analogous result for locally convex real or complex continuous inverse algebras carry over to the present situation. □

6 Mappings between direct sums

Throughout this section, $(\mathbb{K}, |.|)$ denotes a valued field. We study differentiability properties of certain mappings between open subsets of direct sums of topological $\mathbb{K}$-vector spaces.

Given a real number $\varepsilon > 0$, we abbreviate $B_\varepsilon(0) := B^{\mathbb{K}}_\varepsilon(0) = \{x \in \mathbb{K} \colon |x| < \varepsilon\}$. We recall that a subset $U \subseteq E$ of a $\mathbb{K}$-vector space E is called *balanced* if $tU \subseteq U$ for all $t \in \mathbb{K}$ such that $|t| \leq 1$. It is called *absorbing* if, for $x \in E$, there exists $\varepsilon > 0$ such that $B_\varepsilon(0) \cdot x \subseteq U$ (see [12], Ch. I, §1, no. 5).

6.1 Let $(E_i)_{i\in I}$ be a family of topological $\mathbb{K}$-vector spaces, and $E := \bigoplus_{i\in I} E_i$ be its vector space direct sum. Let $\mathcal{F}$ be the set of all sets U of the form

$$U := \bigoplus\nolimits_{i\in I} U_i := E \cap \prod\nolimits_{i\in I} U_i$$

where U_i is an open, balanced zero-neighbourhood in E_i. Then apparently every $U \in \mathcal{F}$ is a balanced and absorbing subset of E, and $tU \in \mathcal{F}$ for each $t \in \mathbb{K}^\times$. It is also easy to find $V \in \mathcal{F}$ such that $V + V \subseteq U$. As a consequence, there is a unique topology on E turning E into a topological $\mathbb{K}$-vector space, and such that $\mathcal{F}$ is a basis for the filter of zero-neighbourhoods of E (see [12], Ch. I, §1, no. 5, Prop. 4). Since $\bigcap \mathcal{F} = \{0\}$, this vector topology is Hausdorff.

6.2 Let $x = (x_i)_{i\in I} \in E$, and suppose that U_i is an open neighbourhood of x_i in E_i, for all $i \in I$. Then $U = \bigoplus_{i\in I} U_i := E \cap \prod_{i\in I} U_i$ is an open neighbourhood of x in E. In fact, let $y = (y_i)_{i\in I} \in U$. Then U_i being a neighbourhood of y_i in E_i, there exists a balanced, open zero-neighbourhood V_i in E_i such that $y_i + V_i \subseteq U_i$. Then $V := \bigoplus_{i\in I} V_i \in \mathcal{F}$, and thus $y + V \subseteq U$ shows that U is a neighbourhood of y. We have shown that U is open.

In the preceding situation, we call U a *box neighbourhood* of x. Accordingly, the topology on E just defined will be called the *box topology*. In this article, direct sums shall always be equipped with the box topology.

6.3 It is obvious from the definition that the box topology on $E = \bigoplus_{i\in I} E_i$ is finer than the topology induced by the product topology on $\prod_{i\in I} E_i$. It is also obvious that the direct sum E induces the product topology on $\prod_{i\in F} E_i = \bigoplus_{i\in F} E_i \subseteq E$, for each finite subset $F \subseteq I$.

6.4 Note that if $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and each E_i is locally convex, then also $\bigoplus_{i\in I} E_i$ is locally convex, because $\bigoplus_{i\in I} U_i$ is convex for any family $(U_i)_{i\in I}$ of convex open 0-neighbourhoods $U_i \subseteq E_i$. Likewise, if $\mathbb{K}$ is an ultrametric field with valuation ring $\mathbb{O}$ and each E_i is locally convex (see **1.2**), then $\bigoplus_{i\in I} E_i$ is locally convex, because $\bigoplus_{i\in I} U_i$ is an open $\mathbb{O}$-submodule of $\bigoplus_{i\in I} E_i$ for any family $(U_i)_{i\in I}$ of open $\mathbb{O}$-submodules $U_i \subseteq E_i$.

6.5 We claim that E, equipped with the box topology, is the direct sum of the family $(E_i)_{i\in I}$ in the category of topological $\mathbb{K}$-vector spaces, *provided that I is countable.* Indeed, this assertion is trivial if I is finite. Otherwise, we may assume that $I = \mathbb{N}$. In this case, suppose that F is a topological $\mathbb{K}$-vector space and $\lambda_n \colon E_n \to F$ a continuous linear mapping for each $n \in \mathbb{N}$. As $E = \bigoplus_{n\in\mathbb{N}} E_n$ as a $\mathbb{K}$-vector space, there is a uniquely determined $\mathbb{K}$-linear map $\lambda \colon E \to F$ such that $\lambda|_{E_n} = \lambda_n$ for each $n \in \mathbb{N}$. Let V_0 be a zero-neighbourhood in F. Inductively, we find a sequence $(V_n)_{n\in\mathbb{N}}$ of open zero-neighbourhoods $V_n \subseteq F$ such that $V_n + V_n \subseteq V_{n-1}$ for all $n \in \mathbb{N}$. Then $U := \bigoplus_{n\in\mathbb{N}} \lambda_n^{-1}(V_n)$ is an open zero-neighbourhood in E such that $\lambda(U) \subseteq \sum_{n\in\mathbb{N}} V_n \subseteq V_0$. We deduce that λ is continuous.

We are primarily interested in the case of countable direct sums, but our arguments will work more generally.

Remark 6.6 If I is uncountable, then the box topology on E need not make E the direct sum of the family $(E_i)_{i\in I}$ in the category of topological $\mathbb{K}$-vector spaces. For example, if $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$) and $(E_i)_{i\in I}$ is an uncountable family of non-zero locally convex topological $\mathbb{K}$-vector spaces, then the locally convex direct sum topology is easily seen to be properly finer than the box topology (since this is so for $\mathbb{R}^{(I)}$).[8]

Remark 6.7 If $(E_i)_{i\in I}$ is any family of locally convex topological vector spaces over an ultrametric field $\mathbb{K}$, then $E := \bigoplus_{i\in I} E_i$, equipped with the box topology, is locally convex (see **6.4**), and it is the direct sum of the family $(E_i)_{i\in I}$ in the category of locally convex topological $\mathbb{K}$-vector spaces. To see this, let $M \subseteq E$ be an $\mathbb{O}$-submodule such that $M_i := M \cap E_i$ is open in E_i for each i. Then $\bigoplus_{i\in I} M_i$ is a box neighbourhood of 0 which is contained in M as M is an $\mathbb{O}$-submodule (and thus an additive subgroup) of E. Consequently, M is open in E. Therefore the box topology is the finest locally convex vector topology on the direct sum E which makes all of the inclusion maps $E_i \to E$ continuous. Hence E, with the box topology, has the universal property of the locally convex direct sum: A linear map $f : E \to F$ in a locally convex space F is continuous if and only if $f|_{E_i} \colon E_i \to F$ is continuous for each $i \in I$.

It is our goal now to explore differentiability properties of mappings between direct sums. Our discussions will hinge on symmetry properties of the maps $f^{[k]}$. In order to formulate these symmetry properties conveniently, we re-order the arguments of $f^{[k]} \colon U^{[k]} \to F$, by grouping the variables in E together on the one hand, on the other hand those in $\mathbb{K}$.

[8] Note that the addition map $\mathbb{R}^{(I)} \to \mathbb{R}$, $(r_i)_{i\in I} \mapsto \sum_{i\in I} r_i$ is discontinuous with respect to the box topology, if I is uncountable.

Given topological $\mathbb{K}$-vector spaces E and F and a C^r-map $f\colon U \to F$ defined on an open subset of E, we let $U^{\{0\}} := U$, $f^{\{0\}} := f$, $U^{\{1\}} := U^{[1]}$, $f^{\{1\}} := f^{[1]}$ and define mappings $f^{\{k+1\}}\colon U^{\{k+1\}} \to F$ for $k \in \mathbb{N}$, $k < r$ on the sets

$$U^{\{k+1\}} := \{(x,y,u,v,t) \in E^{2^k} \times E^{2^k} \times \mathbb{K}^{2^k-1} \times \mathbb{K}^{2^k-1} \times \mathbb{K}\colon (x,u,y,v,t) \in (U^{\{k\}})^{[1]}\}$$

inductively via

$$f^{\{k+1\}}(x,y,u,v,t) := (f^{\{k\}})^{[1]}(x,u,y,v,t)\,.$$

Lemma 6.8 *Given $k \in \mathbb{N}$, there exist $\ell \in \mathbb{N}$, $i_\nu \in \mathbb{N}_0$ for $\nu = 1,\ldots,2^k$ and $j_\mu \in \mathbb{N}_0$ for $\mu = 1,\ldots,2^k-1$ with the following properties:*

(a) *Given an open subset U of a topological $\mathbb{K}$-vector space E, $x = (x_1,\ldots,x_{2^k}) \in E^{2^k}$, $p = (p_1,\ldots,p_{2^k-1}) \in \mathbb{K}^{2^k-1}$ and $t \in \mathbb{K}^\times$, we have $(x,tp) \in U^{\{k\}}$ if and only if*

$$(t^{i_1}x_1,\ldots,t^{i_{2^k}}x_{2^k},t^{-j_1}p_1,\ldots,t^{-j_{2^k-1}}p_{2^k-1}) \in U^{\{k\}}\,.$$

(b) *For any topological $\mathbb{K}$-vector spaces E, F, any C^k-map $f\colon U \to F$ defined on an open subset of E, and each $(x,p,t) \in E^{2^k} \times \mathbb{K}^{2^k-1} \times \mathbb{K}^\times$ such that $(x,tp) \in U^{\{k\}}$, we have:*

$$f^{\{k\}}(x,tp) = t^{-\ell}\cdot f^{\{k\}}(t^{i_1}x_1,\ldots,t^{i_{2^k}}x_{2^k},t^{-j_1}p_1,\ldots,t^{-j_{2^k-1}}p_{2^k-1})\,. \qquad (16)$$

Proof. The proof is by induction on $k \in \mathbb{N}$. If $k = 1$, let $x_1, x_2 \in E$, $p \in \mathbb{K}$, $t \in \mathbb{K}^\times$. Then $(x_1,x_2,tp) \in U^{[1]}$ if and only if $x_1 \in U$ and $x_1 + (tp)x_2 = x_1 + p(tx_2) \in U$, which holds precisely if $(x_1,tx_2,p) \in U^{[1]}$. Assume that $(x_1,x_2,tp) \in U^{[1]}$. If $p \neq 0$, we have

$$f^{[1]}(x_1,x_2,tp) = \tfrac{1}{tp}(f(x_1+tpx_2)-f(x_1)) = \tfrac{1}{t}f^{[1]}(x_1,tx_2,p).$$

By continuity, $f^{[1]}(x_1,x_2,tp) = \frac{1}{t}f^{[1]}(x_1,tx_2,p)$ then also holds if $p = 0$.

Induction step. Suppose the lemma is correct for a certain $k \in \mathbb{N}_0$; let ℓ and i_ν, j_μ be as described in the lemma. Suppose further that $f\colon U \to F$ is of class C^{k+1}. Let $x,y \in E^{2^k}$, $u,v \in \mathbb{K}^{2^k-1}$, $s \in \mathbb{K}$ and $t \in \mathbb{K}^\times$ such that $(x,y,tu,tv,ts) \in U^{\{k+1\}}$. If $s \neq 0$, we calculate

$$\begin{aligned}
&f^{\{k+1\}}(x,y,tu,tv,ts) &&(17)\\
&= (f^{\{k\}})^{[1]}(x,tu;y,tv;ts)\\
&= \tfrac{1}{ts}\big(f^{\{k\}}((x,tu)+ts(y,tv)) - f^{\{k\}}(x,tu)\big)\\
&= \tfrac{1}{ts}\big(f^{\{k\}}(x+tsy,t^2(\tfrac{1}{t}u+sv)) - f^{\{k\}}(x,t^2(\tfrac{1}{t}u))\big)\\
&= \tfrac{1}{ts\cdot t^{2\ell}}[f^{\{k\}}(t^{2i_1}x_1+t^{2i_1+1}sy_1,\ldots,t^{2i_{2^k}}x_{2^k}+t^{2i_{2^k}+1}sy_{2^k},\ t^{-2j_1-1}u_1+t^{-2j_1}sv_1,\\
&\qquad\qquad \ldots,t^{-2j_{2^k-1}-1}u_{2^k-1}+t^{-2j_{2^k-1}}sv_{2^k-1})\\
&\quad -f^{\{k\}}(t^{2i_1}x_1,\ldots,t^{2i_{2^k}}x_{2^k},t^{-2j_1-1}u_1,\ldots,t^{-2j_{2^k-1}-1}u_{2^k-1})]\\
&= \tfrac{1}{t^{2\ell+1}}(f^{\{k\}})^{[1]}(t^{2i_1}x_1,\ldots,t^{2i_{2^k}}x_{2^k},t^{-2j_1-1}u_1,\ldots,t^{-2j_{2^k-1}-1}u_{2^k-1},\\
&\qquad\qquad t^{2i_1+1}y_1,\ldots,t^{2i_{2^k}+1}y_{2^k},t^{-2j_1}v_1,\ldots,t^{-2j_{2^k-1}}v_{2^k-1},\ s)\\
&= \tfrac{1}{t^{2\ell+1}}f^{\{k+1\}}(t^{2i_1}x_1,\ldots,t^{2i_{2^k}}x_{2^k},t^{2i_1+1}y_1,\ldots,t^{2i_{2^k}+1}y_{2^k},\\
&\qquad\qquad t^{-2j_1-1}u_1,\ldots,t^{-2j_{2^k-1}-1}u_{2^k-1},t^{-2j_1}v_1,\ldots,t^{-2j_{2^k-1}}v_{2^k-1},\ s)\,,
\end{aligned}$$

where the calculation shows that the argument of the function in the last line is in $U^{\{k+1\}}$. Here, the induction hypothesis was used to obtain the fourth equality in (17). If $s=0$, there exists a zero-neighbourhood S in $\mathbb{K}$ such that $(x,y,tu,tv,ts')\in U^{\{k+1\}}$ for all $s'\in S$. There exists $s'\in S\setminus\{0\}$. By the above, we then have

$$\begin{aligned}(t^{2i_1}x_1,\ldots,t^{2i_{2^k}}x_{2^k},t^{-2j_1-1}u_1,\ldots,t^{-2j_{2^k-1}-1}u_{2^k-1},t^{2i_1+1}y_1,\ldots,t^{2i_{2^k}+1}y_{2^k},\\ t^{-2j_1}v_1,\ldots,t^{-2j_{2^k-1}}v_{2^k-1},s')\in (U^{\{k\}})^{[1]}\end{aligned}$$

and thus $(t^{2i_1}x_1,\ldots,t^{2i_{2^k}}x_{2^k},t^{-2j_1-1}u_1,\ldots,t^{-2j_{2^k-1}-1}u_{2^k-1})\in U^{\{k\}}$, entailing that

$$\begin{aligned}(t^{2i_1}x_1,\ldots,t^{2i_{2^k}}x_{2^k},t^{-2j_1-1}u_1,\ldots,t^{-2j_{2^k-1}-1}u_{2^k-1},t^{2i_1+1}y_1,\ldots,t^{2i_{2^k}+1}y_{2^k},\\ t^{-2j_1}v_1,\ldots,t^{-2j_{2^k-1}}v_{2^k-1},0)\in (U^{\{k\}})^{[1]}\end{aligned}$$

and hence

$$\begin{aligned}(t^{2i_1}x_1,\ldots,t^{2i_{2^k}}x_{2^k},t^{2i_1+1}y_1,\ldots,t^{2i_{2^k}+1}y_{2^k},t^{-2j_1-1}u_1,\ldots,t^{-2j_{2^k-1}-1}u_{2^k-1},\\ t^{-2j_1}v_1,\ldots,t^{-2j_{2^k-1}}v_{2^k-1},\ s)\in U^{\{k+1\}}\end{aligned}\tag{18}$$

with $s=0$. By continuity, the first and final term in display (17) also coincide when $s=0$. To complete the proof of (a), assume, conversely, that $x,y\in E^{2^k}$, $u,v\in\mathbb{K}^{2^k-1}$, $s\in\mathbb{K}$ and $t\in\mathbb{K}^\times$ are given such that (18) holds. If $s\neq 0$, exploiting the induction hypothesis we can go backwards from bottom to top in the display (17), and deduce that $(x,y,tu,tv,ts)\in U^{\{k+1\}}$. Arguing as above, we see that this conclusion remains valid when $s=0$. Thus (a) and (b) are established also for k replaced with $k+1$. □

The proof shows that we can achieve $\ell=2^k-1$ here.

We are now ready for the main result of this section.

Proposition 6.9 *Suppose that $(E_i)_{i\in I}$ and $(F_i)_{i\in I}$ are families of topological $\mathbb{K}$-vector spaces indexed by a set I. Let $k\in\mathbb{N}_0\cup\{\infty\}$, and suppose that $f_i\colon U_i\to F_i$ is a mapping of class C^k for $i\in I$, defined on an open non-empty subset U_i of E_i. Suppose that there is a finite subset $J\subseteq I$ such that $0\in U_i$ and $f_i(0)=0$, for all $i\in I\setminus J$. Then $U:=\bigoplus_{i\in I}U_i$ is an open subset of $E:=\bigoplus_{i\in I}E_i$, and*

$$f:=\bigoplus_{i\in I}f_i\colon U\to F,\qquad f((x_i)_{i\in I}):=(f_i(x_i))_{i\in I}$$

is a mapping of class C^k into $F:=\bigoplus_{i\in I}F_i$. For each $j\in\mathbb{N}$ such that $j\leq k$, identifying E^{2^j} with $\bigoplus_{i\in I}E_i^{2^j}$ in the natural way, we have

$$U^{\{j\}}=\{((x_i)_{i\in I},p)\in E^{2^j}\times\mathbb{K}^{2^j-1}\colon\ (\forall i\in I)\ (x_i,p)\in U_i^{\{j\}}\},\quad \textit{and}\tag{19}$$

$$f^{\{j\}}((x_i)_{i\in I},p)=(f_i^{\{j\}}(x_i,p))_{i\in I}\,.$$

Proof. We may assume that $k \in \mathbb{N}_0$; the proof is by induction on k.

The case $k = 0$. Let $x = (x_i)_{i\in I} \in U$ and V be a neighbourhood of $f(x)$. Then V contains a box-neighbourhood $B = \bigoplus_{i\in I} V_i$ of $f(x)$, where V_i is an open neighbourhood of $f_i(x_i)$. As $f^{-1}(V)$ contains the box-neighbourhood $f^{-1}(B) = \bigoplus_{i\in I} f_i^{-1}(V_i)$ of x, the set $f^{-1}(V)$ is a neighbourhood of x. We have shown that f is continuous at x.

Induction step. Suppose the assertion holds for a given $k \in \mathbb{N}_0$, and suppose that each f_i is a mapping of class C^{k+1}. Then f is of class C^k, and $f^{\{k\}}((x_i)_{i\in I}, p) = (f_i^{\{k\}}(x_i, p))_{i\in I}$. Equation (19) holds for $j \leq k$ by induction and thus also for $j = k+1$, as an immediate consequence of the definitions. We claim that $f^{\{k\}}$ is of class C^1. Let $x = (x_i)_{i\in I}$, $y = (y_i)_{i\in I} \in E^{2^k} \cong \bigoplus_{i\in I} E_i^{2^k}$, $u, v \in \mathbb{K}^{2^k-1}$ and $t \in \mathbb{K}$ such that $(x, u, y, v, t) \in (U^{\{k\}})^{[1]}$. If $t \neq 0$, we have, by induction,

$$\begin{aligned} \tfrac{1}{t}(f^{\{k\}}(x+ty, u+tv) - f^{\{k\}}(x,u)) &= (\tfrac{1}{t} f_i^{\{k\}}(x_i + ty_i, u+tv) - f_i^{\{k\}}(x_i, u)))_{i\in I} \\ &= (f_i^{\{k+1\}}(x_i, y_i, u, v, t))_{i\in I}\,. \end{aligned}$$

Thus $f^{\{k\}}$ will be of class C^1 if we can show that the mapping

$$(U^{\{k\}})^{[1]} \to F, \quad (x, u, y, v, t) \mapsto (f_i^{\{k+1\}}(x_i, y_i, u, v, t))_{i\in I}$$

is continuous, or, equivalently, that

$$g\colon U^{\{k+1\}} \to F, \quad (x, y, u, v, t) \mapsto (f_i^{\{k+1\}}(x_i, y_i, u, v, t))_{i\in I}$$

is continuous—this is our goal now. We have $\{0\} \times \mathbb{K}^{2^j-1} \subseteq U_i^{\{j\}}$ for all $i \in I \setminus J$ and all $j \in \mathbb{N}$ such that $j \leq k+1$, and

$$f_i^{\{j\}}(0, p) = 0 \quad \text{for all } p \in \mathbb{K}^{2^j-1}, \tag{20}$$

by a simple induction. Let $\bar{x} = (\bar{x}_i)_{i\in I} \in E^{2^{k+1}} \cong \bigoplus_{i\in I} E_i^{2^{k+1}}$, $\bar{p} = (\bar{p}_\nu)_{\nu=1}^{2^{k+1}-1} \in \mathbb{K}^{2^{k+1}-1}$ such that $(\bar{x}, \bar{p}) \in U^{\{k+1\}}$. Pick a real number $r > \|\bar{p}\|_\infty$. There is a finite subset $J_0 \subseteq I$ such that $J \subseteq J_0$ and such that $\bar{x}_i = 0$ for all $i \in I \setminus J_0$. Let W be an open neighbourhood of $g(\bar{x}, \bar{p})$ in F; we may assume that $W = \bigoplus_{i\in I} W_i$, where W_i is an open neighbourhood of $f_i^{\{k+1\}}(\bar{x}_i, \bar{p})$ in F_i. For $i \in I \setminus J_0$, we may assume that the zero-neighbourhood W_i is balanced.

Let $\ell \in \mathbb{N}$, $i_\mu \in \mathbb{N}_0$ for $\mu = 1, \ldots, 2^{k+1}$ and $j_\nu \in \mathbb{N}_0$ for $\nu = 1, \ldots, 2^{k+1}-1$ be as in the C^{k+1}-case of Lemma 6.8.

For each $i \in I \setminus J_0$, there exists $\varepsilon_i > 0$ and an open balanced zero-neighbourhood $V_i \subseteq E_i$ such that

$$f_i^{\{k+1\}}(V_i^{2^{k+1}} \times B_{\varepsilon_i}(0)^{2^{k+1}-1}) \subseteq W_i\,.$$

There exists $\tau_i \in \mathbb{K}^\times$ such that $|\tau_i| > \max\{1, \frac{r}{\varepsilon_i}\}$; set $A_i := \prod_{\mu=1}^{2^{k+1}} \tau_i^{-i_\mu} V_i$. Holding $i \in I \setminus J_0$ fixed for the moment, let us write $\tau := \tau_i$, for convenience. For all $x = (x_\mu)_{\mu=1}^{2^{k+1}} \in A_i$ and $p = (p_\nu)_{\nu=1}^{2^{k+1}-1} \in B_r(0)^{2^{k+1}-1}$, we have $\tau^{i_\mu} x_\mu \in V_i$ for $\mu = 1, \ldots, 2^{k+1}$ and $|\tau^{-j_\nu-1} p_\nu| =$

$|\tau|^{-j_\nu-1}\cdot|p_\nu|<\frac{r}{|\tau|}<\varepsilon_i$ for $\nu=1,\ldots,2^{k+1}-1$, *i.e.*, $\tau^{-j_\nu-1}p_\nu\in B_{\varepsilon_i}(0)$. Thus Lemma 6.8 (a) shows that $(x,p)\in U_i^{\{k+1\}}$ and

$$\begin{aligned}
&f_i^{\{k+1\}}(x,p)-f_i^{\{k+1\}}(\overline{x}_i,\overline{p})\\
&= f_i^{\{k+1\}}(x,p)=f_i^{\{k+1\}}(x,\tau(\tau^{-1}p))\\
&= \tfrac{1}{\tau^\ell}f_i^{\{k+1\}}(\tau^{i_1}x_1,\ldots,\tau^{i_{2^{k+1}}}x_{2^{k+1}},\tau^{-j_1-1}p_1,\ldots,\tau^{-j_{2^{k+1}-1}-1}p_{2^{k+1}-1})\in\tfrac{1}{\tau^\ell}W_i\subseteq W_i\,,
\end{aligned}$$

using (20) to pass to the second line and Lemma 6.8 (b) to pass to the third. For each $i\in J_0$, on the other hand, by continuity of $f_i^{\{k+1\}}$ there exists an open neighbourhood $A_i\subseteq E_i$ of $\bar{x}_i$ and an open neighbourhood Z_i of $\overline{p}$ in $\mathbb{K}^{2^{k+1}-1}$ such that $A_i\times Z_i\subseteq U_i^{\{k+1\}}$ and $f_i^{\{k+1\}}(A_i\times Z_i)\subseteq W_i$. Then $Z:=B_r(0)^{2^{k+1}-1}\cap\bigcap_{i\in J_0}Z_i$ is an open neighbourhood of $\bar{p}$ in $\mathbb{K}^{2^{k+1}-1}$. Let $A:=\bigoplus_{i\in I}A_i$. Then $A\times Z$ is an open neighbourhood of $(\bar{x},\bar{p})$ in $U^{\{k+1\}}$ such that

$$g(x,p)\in W\quad\text{for all }\;(x,p)\in A\times Z.$$

We have shown that g is continuous at $(\bar{x},\bar{p})$. Thus $f^{\{k\}}$ is of class C^1 and hence also $f^{[k]}$ is of class C^1 (by the Chain Rule). Hence f is of class C^{k+1}. Furthermore, $f^{\{k+1\}}=g$ is of the asserted form. □

Results analogous to Proposition 6.9 for mappings between locally convex direct sums of real or complex locally convex spaces have first been established in [22]; the proofs are considerably easier in that case.

Analogues for functions involving parameters

When the ground field $\mathbb{K}$ is locally compact, Proposition 6.9 can be generalized to functions involving parameters (and its proof simplifies substantially).

Proposition 6.10 *Let $(\mathbb{K},|.|)$ be a valued field, $P\neq\emptyset$ be a locally compact topological space, $(E_i)_{i\in I}$ and $(F_i)_{i\in I}$ be families of topological $\mathbb{K}$-vector spaces indexed by a set I, and $(f_i)_{i\in I}$ be a family of continuous mappings $f_i\colon U_i\times P\to F_i$, where U_i is a non-empty open subset of E_i. Suppose that there is a finite subset $J\subseteq I$ such that $0\in U_i$ and $f_i(0,p)=0$, for all $i\in I\setminus J$ and $p\in P$. Then $U:=\bigoplus_{i\in I}U_i$ is an open subset of $E:=\bigoplus_{i\in I}E_i$, and*

$$f:=U\times P\to F,\qquad f((x_i)_{i\in I},p):=(f_i(x_i,p))_{i\in I}$$

is a continuous map into $F:=\bigoplus_{i\in I}F_i$. If $\mathbb{K}$ is locally compact here, P an open subset of a finite-dimensional $\mathbb{K}$-vector space Z, and if there exists $k\in\mathbb{N}_0\cup\{\infty\}$ such that f_i is of class C^k for all $i\in I$, then also f is of class C^k.

Proof. We may assume that $k\in\mathbb{N}_0$; the proof is by induction on k.

The case $k=0$. Let $x=(x_i)_{i\in I}\in U$, $p\in P$, and V be a neighbourhood of $f(x,p)$ in F. Then V contains a box-neighbourhood $B=\bigoplus_{i\in I}V_i$ of $f(x,p)$, where V_i is an open

neighbourhood of $f_i(x_i,p)$ in F_i. There is a finite subset $J_0 \subseteq I$ such that $J \subseteq J_0$ and such that $x_i = 0$ for all $i \in I \setminus J_0$. For each $i \in J_0$, we find a compact neighbourhood K_i of p in P and an open neighbourhood $W_i \subseteq U_i$ of x_i such that $f_i(W_i \times K_i) \subseteq V_i$. Then $K := \bigcap_{i \in J_0} K_i$ is a compact neighbourhood of p in P. For each $i \in I \setminus J_0$, we have

$$f_i(\{0\} \times K) = \{0\} \subseteq V_i\,.$$

Using the compactness of K, we therefore find an open zero-neighbourhood $W_i \subseteq E_i$ such that $f_i(W_i \times K) \subseteq V_i$. Then $W := \bigoplus_{i \in I} W_i \subseteq U$ is an open neighbourhood of x, and $f(W \times K) \subseteq B$ since $f_i(W_i \times K) \subseteq V_i$ for all i. Thus f is continuous.

Induction step. Let $\mathbb{K}$ be locally compact now, $k \in \mathbb{N}$, and suppose that the assertion of the proposition holds when k is replaced with $k-1$. Let $P \subseteq Z$ and C^k-maps $f_i \colon U_i \times P \to F_i$ be given. Then $f \colon U \times P \to F$ is a C^{k-1}-map (and thus continuous), by induction. As

$$(U \times P)^{[1]} = \big\{(x,p,y,q,t) \in (E \times Z)^2 \times \mathbb{K} \colon (x_i,p,y_i,q,t) \in (U_i \times P)^{[1]} \text{ for all } i \in I\,\big\}$$

clearly (where $x = (x_i)_{i \in I}$, $y = (y_i)_{i \in I}$), we can define a mapping

$$g \colon (U \times P)^{[1]} \to F, \quad g(x,p,y,q,t) := (f_i^{[1]}(x_i,p,y_i,q,t))_{i \in I}\,.$$

Let us show that f is C^1, with $f^{[1]} = g$ of class C^{k-1}. Since

$$\tfrac{1}{t}(f(x+ty,p+tq) - f(x,p)) = \big(\tfrac{1}{t}(f_i(x_i+ty_i,p+tq) - f_i(x_i,p))\big)_{i \in I} = g(x,p,y,q,t)$$

for all $(x,p,y,q,t) \in (U \times P)^{[1]}$ such that $t \neq 0$, it suffices to show that g is of class C^{k-1}. Now f being of class C^{k-1}, the map g is C^{k-1} on the set $\{(x,p,y,q,t) \in (U \times P)^{[1]} \colon t \neq 0\}$. It therefore only remains to show that g is C^k on some open neighbourhood of $(\bar{x},\bar{p},\bar{y},\bar{q},0)$, for all $\bar{x} = (\bar{x}_i) \in U$, $\bar{p} \in P$, $\bar{y} = (\bar{y}_i) \in E$, and $\bar{q} \in Z$. For each $i \in I$, we find an open, balanced zero-neighbourhood $W_i \subseteq E_i$ such that $\bar{x}_i + W_i + W_i \subseteq U_i$. Then $A_i := \bar{x}_i + W_i \subseteq U_i$. Since $\bar{y}_i = 0$ for all but finitely many i, we find $r \in \,]0,1]$ such that $t\bar{y}_i \in W_i$ for all $i \in I$ and $t \in \mathbb{K}$ such that $|t| \leq r$. Pick $\rho \in \mathbb{K}^\times$ such that $|\rho| \leq r$; then $B_i := \rho^{-1} W_i$ is an open neighbourhood of $\bar{y}_i$, for all $i \in I$. There are $s \in \,]0, r|\rho|]$ and open neighbourhoods $R \subseteq P$ of $\bar{p}$ and $S \subseteq Z$ of $\bar{q}$, such that $R \times S \times B_s(0) \subseteq P^{[1]}$, where $B_s(0) \subseteq \mathbb{K}$. Then also $A_i \times B_i \times B_s(0) \subseteq U_i^{[1]}$ for each $i \in I$ and hence

$$A \times R \times B \times S \times B_s(0) \subseteq (U \times P)^{[1]}\,,$$

where $A := \bigoplus_{i \in I} A_i \subseteq E$ and $B := \bigoplus_{i \in I} B_i \subseteq E$. Let $Q := R \times S \times B_s(0)$; then

$$h_i \colon (A_i \times B_i) \times Q \to F, \quad h_i(x_i,y_i,p,q,t) := f_i^{[1]}(x_i,p,y_i,q,t)$$

is a C^{k-1}-map, for each $i \in I$. Furthermore, $h_i|_{\{0\} \times Q} = 0$ for all $i \in I \setminus J$. Define

$$h \colon (A \times B) \times Q \to F, \quad h(x,y,p,q,t) := (h_i(x_i,y_i,p,q,t))_{i \in I} = g(x,p,y,q,t)\,.$$

Here $A \times B \subseteq E \times E \cong \bigoplus_{i \in I}(E_i \times E_i)$, and Q is an open subset of the finite-dimensional $\mathbb{K}$-vector space $Z \times Z \times \mathbb{K}$, which is locally compact since so is $\mathbb{K}$. By the induction hypothesis, h is of class C^{k-1}. Hence g is C^{k-1} on the open neighbourhood $A \times R \times B \times S \times B_s(0)$ of $(\bar{x},\bar{p},\bar{y},\bar{q},0)$, which completes the proof. $\square$

Remark 6.11 Let $\mathbb{K}$ be $\mathbb{R}$ or a local field, $n \in \mathbb{N}$, and $G := \mathrm{Diff}_c^\infty(\mathbb{K}^n) \rtimes \mathrm{Aff}(\mathbb{K}^n)$ be the group of all smooth diffeomorphisms of $\mathbb{K}^n$ which coincide with an affine isomorphism of $\mathbb{K}^n$ outside some compact set. Using Proposition 6.10, it is possible to make G a $\mathbb{K}$-Lie group modeled on the topological $\mathbb{K}$-vector space $C_c^\infty(\mathbb{K}^n, \mathbb{K}^n) \times \mathrm{aff}(\mathbb{K}^n)$. We omit the proof. A more profound application of Proposition 6.10 will be given in Theorem F.23 below.

7 Weak direct products of Lie groups

The considerations in Section 6 make it possible to construct Lie group structures on weak direct products of Lie groups.

Proposition 7.1 *Let $(G_i)_{i\in I}$ be a family of Lie groups over a valued field $(\mathbb{K}, |.|)$. Then there exists a unique $\mathbb{K}$-Lie group structure on*

$$\prod\nolimits^*_{i\in I} G_i := \{(g_i)_{i\in I} \in \prod\nolimits_{i\in I} G_i \colon \; g_i = 1 \textit{ for all but finitely many } i\,\},$$

modeled on $\bigoplus_{i\in I} L(G_i)$, equipped with the box topology, such that, for certain charts $\kappa_i \colon U_i \to V_i \subseteq L(G_i)$ of G_i defined on an identity neighbourhood $U_i \subseteq G_i$ and taking 1 to 0, the mapping

$$\bigoplus\nolimits_{i\in I} V_i \to \prod\nolimits^*_{i\in I} G_i, \quad (x_i)_{i\in I} \mapsto (\kappa_i^{-1}(x_i))_{i\in I}$$

is a diffeomorphism of smooth $\mathbb{K}$-manifolds onto an open subset of $\prod^*_{i\in I} G_i$.

Proof. Using Proposition 6.9 instead of [22, Prop. 7.1], the proof of [22, Prop. 7.3] (devoted to weak direct products of real or complex Lie groups modeled on locally convex spaces) carries over to the present situation (see [27, Thm. 18.1] for further details). □

The following observations are immediate from the construction of the Lie group structure on weak direct products and obvious properties of direct sums of topological vector spaces:

Lemma 7.2 *Let $\mathbb{K}$ be a valued field.*

(a) *If $(G_i)_{i\in I}$ is a family of $\mathbb{K}$-Lie groups and $(H_i)_{i\in I}$ a family of open subgroups $H_i \subseteq G_i$, then $\prod^*_{i\in I} H_i$ is an open subset of $\prod^*_{i\in I} G_i$. The smooth manifold structure making $\prod^*_{i\in I} H_i$ an open submanifold of $\prod^*_{i\in I} G_i$ and the manifold structure on the weak direct product of Lie groups $\prod^*_{i\in I} H_i$ coincide.*

(b) *Assume that I is a set, J_i a finite set for each $i \in I$, and $K := \{(i,j) \colon i \in I, j \in J_i\}$. Let $(G_{ij})_{(i,j)\in K}$ be a family of $\mathbb{K}$-Lie groups. Then the mapping*

$$\prod\nolimits^*_{(i,k)\in K} G_{ij} \to \prod\nolimits^*_{i\in I}\Big(\prod\nolimits_{j\in J_i} G_{ij}\Big), \quad (g_{ij})_{(i,j)\in K} \mapsto \big((g_{ij})_{j\in J_i}\big)_{i\in I}$$

is an isomorphism of $\mathbb{K}$-Lie groups.

(c) *If $(G_i)_{i\in I}$ and $(H_j)_{j\in J}$ are families of $\mathbb{K}$-Lie groups, $\pi \colon J \to I$ is a bijection and $\beta_j \colon G_{\pi(j)} \to H_j$ an isomorphism of $\mathbb{K}$-Lie groups for each $j \in J$, then also the map*

$$\prod\nolimits^*_{i\in I} G_i \to \prod\nolimits^*_{j\in J} H_j\,, \quad (g_i)_{i\in I} \mapsto \big(\beta_j(g_{\pi(j)})\big)_{j\in J}$$

is an isomorphism of $\mathbb{K}$-Lie groups. □

8 Spaces of test functions and mappings between them

In this section (and in Section 10), we study differentiability properties of mappings between spaces of vector-valued test functions on paracompact finite-dimensional manifolds over locally compact ground fields. First, we collect some properties of such manifolds.

Paracompact finite-dimensional manifolds over locally compact fields

Throughout this subsection, $\mathbb{F}$ is a (non-discrete), locally compact topological field, and $r \in \mathbb{N}_0 \cup \{\infty\}$.

Paracompact manifolds over locally compact fields are amenable to investigation due to the following well-known fact (see [16, Thm. 5.1.27]): For every paracompact, locally compact topological space X, there exists a cover $(X_i)_{i\in I}$ of X by mutually disjoint, σ-compact, open (and closed) subsets $X_i \subseteq X$ (and thus $X = \coprod_{i\in I} X_i$). As a special case, we obtain:

Lemma 8.1 *Every paracompact, finite-dimensional $C^r_{\mathbb{F}}$-manifold M is a disjoint union $M = \coprod_{i\in I} M_i$ of a family $(M_i)_{i\in I}$ of σ-compact, open (and closed) submanifolds $M_i \subseteq M$.* $\square$

8.2 If $\mathbb{F}$ is a local field, we fix the following notation: $|.|$ is an ultrametric absolute value on $\mathbb{F}$ defining its topology, $\mathbb{O}$ the maximal compact subring of $\mathbb{F}$, and $\pi \in \mathbb{F}^\times$ a uniformizing element (thus $|\pi| < 1$ and $|\mathbb{F}^\times| = \langle |\pi| \rangle$). Given $d \in \mathbb{N}$, we let $\|\bullet\|_\infty$ be the maximum norm on $\mathbb{F}^d$. Given $a \in \mathbb{K}^d$ and $\varepsilon > 0$, $B_\varepsilon(a) := \{y \in \mathbb{F}^d \colon \|y-a\|_\infty < \varepsilon\}$ denotes the ball with respect to the maximum norm. Then $B := \mathbb{O}^d$ is an open compact $\mathbb{O}$-submodule of $\mathbb{F}^d$, and it is easy to see that each ball $B_\varepsilon(a)$ is of the form $a + \pi^k B$ for some $k \in \mathbb{Z}$ and thus $C^\infty_{\mathbb{F}}$-diffeomorphic to B. If M is a d-dimensional $\mathbb{F}$-manifold of class $C^r_{\mathbb{F}}$, we call an open subset of M a *ball* if it is $C^r_{\mathbb{F}}$-diffeomorphic to B. It is clear that every point $x \in M$ is contained in some ball. To avoid misunderstandings, the balls $B_\varepsilon(a) \subseteq \mathbb{F}^d$ will occasionally be called *metric balls* now.

The following lemma assembles various useful facts concerning paracompact manifolds over local fields (cf. also [50]).

Lemma 8.3 *Let $\mathbb{F}$ be a local field, $r \in \mathbb{N}_0 \cup \{\infty\}$, and M be an $C^r_{\mathbb{F}}$-manifold over $\mathbb{F}$, of positive, finite dimension $d \in \mathbb{N}$. Then the following holds:*

(a) *If M is σ-compact, then M is $C^r_{\mathbb{F}}$-diffeomorphic to an open subset $U \subseteq \mathbb{F}^d$.*

(b) *If M is paracompact, then M is a disjoint union $M = \coprod_{i\in I} B_i$ of a family $(B_i)_{i\in I}$ of compact and open balls $B_i \subseteq M$.*

Proof. (a) Since M is σ-compact, there exists a sequence $(B_k)_{k\in\mathbb{N}}$ of balls covering M. We set $J_1 := \{B_1\}$. Suppose that we have found an open cover J_k of $\bigcup_{j=1}^k B_j$ by disjoint balls for $k = 1, \ldots, n$, such that $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n$. Let $\psi \colon B_{n+1} \to B$ be a $C^r_{\mathbb{F}}$-diffeomorphism onto $B := \mathbb{O}^d$. Then $R := B_{n+1} \setminus (\bigcup J_n) = B_{n+1} \setminus (\bigcup_{k=1}^n B_k)$ is an open, compact subset of B_{n+1} and thus $\psi(R)$ is an open, compact subset of B. As $\psi(R)$ is open and compact,

there exists $\varepsilon \in \,]0,1]$ such that $\psi(R) + B_\varepsilon(0) \subseteq \psi(R)$. Since $B_\varepsilon(0)$ is an open subgroup of the compact additive group B, we deduce that $\psi(R)$ is the disjoint union of a finite number of balls $B_\varepsilon(a_1), \cdots, B_\varepsilon(a_m)$ (i.e., cosets of $B_\varepsilon(0)$), for some $m \in \mathbb{N}_0$ and elements $a_1, \ldots, a_m \in B$. Then $J_{n+1} := J_n \cup \{\psi^{-1}(B_\varepsilon(a_i)) \colon i = 1, \ldots, m\}$ is an open cover of B_{n+1} by mutually disjoint balls, and $J_n \subseteq J_{n+1}$ by definition. Proceeding in this way, we obtain an ascending sequence $J_1 \subseteq J_2 \subseteq \cdots$, where each J_n is an open cover of $\bigcup_{k=1}^n B_k$ by mutually disjoint balls. Thus $J := \bigcup_{k\in\mathbb{N}} J_k$ is a countable cover of M by mutually disjoint balls. Choose an injection $\kappa \colon J \to \mathbb{N}$. For each ball $C_j := j \in J$, there exists a $C^r_{\mathbb{F}}$-diffeomorphism $\phi_j \colon C_j \to \pi^{-\kappa(j)} + B \subseteq \mathbb{F}^d$. Then $U := \bigcup_{j\in J}(\pi^{-\kappa(j)} + B)$ is an open subset of $\mathbb{F}^d$. The union defining U is disjoint, because $|\pi^{-\kappa(j)} + x| = \max\{|\pi^{-\kappa(j)}|, |x|\} = |\pi^{-\kappa(j)}| = |\pi|^{-\kappa(j)}$ for each $j \in J$ and $x \in B$. Hence $\phi := \coprod_{j\in J} \phi_j \colon M \to U$ (the map determined by $\phi|_{C_j} = \phi_j$) is a $C^r_{\mathbb{F}}$-diffeomorphism.

(b) By Lemma 8.1, M is a disjoint union $M = \coprod_{i\in I} M_i$ of σ-compact, open and closed submanifolds M_i. The proof of (a) shows that each M_i is a disjoint union $M_i = \coprod_{j\in J_i} C_{i,j}$ of a countable family $(C_{i,j})_{j\in J_i}$ of balls $C_{i,j} \subseteq M_i$. Set $K := \{(i,j) \colon i \in I, j \in J_i\}$. Then $M = \coprod_{(i,j)\in K} C_{i,j}$ is a disjoint union of balls. □

If U is an open subset of $\mathbb{F}^d$, we can even find partitions into metric balls subordinate to any given open cover:

Lemma 8.4 *Suppose that $\mathbb{F}$ is a local field, $d \in \mathbb{N}$ and $U \subseteq \mathbb{F}^d$ a non-empty, open subset. Let $(U_i)_{i\in I}$ be an open cover of U. Then there exist families $(r_j)_{j\in J}$ and $(a_j)_{j\in J}$ of positive real numbers $r_j > 0$, resp., elements $a_j \in U$, indexed by a countable set J, such that $(B_{r_j}(a_j))_{j\in J}$ is an open cover of U by mutually disjoint sets and furthermore the open cover $(B_{r_j}(a_j))_{j\in J}$ is subordinate to $(U_i)_{i\in I}$, viz. for every $j \in J$, there exists $i(j) \in I$ such that $B_{r_j}(a_j) \subseteq U_{i(j)}$.*

Proof. Since U is σ-compact, we find a sequence $(B_k)_{k\in\mathbb{N}}$ of metric balls covering U and which is subordinate to $(U_i)_{i\in I}$: For each $k \in \mathbb{N}$, there exists $i_k \in I$ such that $B_k \subseteq U_{i_k}$. Adapting the proof of Lemma 8.3 (a) in the obvious way,[9] we arrive at a countable cover $J = \bigcup_{k\in\mathbb{N}} J_k$ of U by mutually disjoint metric balls, such that all balls $C \in J_1$ are subsets of $B_1 \subseteq U_{i(1)}$ and all balls $C \in J_{k+1} \setminus J_k$ are subsets of $B_{k+1} \subseteq U_{i_{k+1}}$. □

Locally finite, relatively compact, open covers can always be thickened.

Lemma 8.5 *Let $\mathbb{F}$ be a locally compact field, M be a paracompact, finite-dimensional $C^r_{\mathbb{F}}$-manifold, and $(U_i)_{i\in I}$ be a locally finite cover of M by relatively compact, open subsets $U_i \subseteq M$. Then there exists a locally finite cover $(\widetilde{U}_i)_{i\in I}$ of M by relatively compact, open subsets $\widetilde{U}_i \subseteq M$, such that for each $i \in I$ the closure $\overline{U}_i$ of U_i in M is contained in $\widetilde{U}_i$.*

Proof. To reduce the assertion to the σ-compact case, we first observe that U_i is σ-compact, for each $i \in I$ (using that $\overline{U}_i$ can be covered by finitely many balls). We now

[9] Thus, we choose each ϕ of the form $\phi(z) = az + b$ with suitable $a \in \mathbb{F}^\times$, $b \in \mathbb{F}^d$.

write $i \sim j$ for $i, j \in I$ if and only if there exists $n \in \mathbb{N}$ and $k_1, \ldots, k_n \in I$ such that $k_1 = i$, $k_n = j$ and $U_{k_\nu} \cap U_{k_{\nu+1}} \neq \emptyset$ for $\nu = 1, \ldots, n-1$. Then $\sim$ is an equivalence relation. Since $(U_i)_{i \in I}$ is a locally finite cover and each U_i is relatively compact, we deduce that each equivalence class $C \in I/\sim\; =: J$ is countable; hence

$$M_C := \bigcup_{i \in C} U_i$$

is a σ-compact open subset of M. By construction, $M = \coprod_{C \in J} M_C$ is a disjoint union of the open (and hence also closed) sets M_C. Since $(U_i)_{i \in C}$ is a countable, locally finite cover of M_C by relatively compact, open sets, it suffices to prove our assertion for countable covers of the M_C's. We may hence assume that M is σ-compact and that I is countable. If I is finite, then M is compact and the assertion is trivial. Thus $I = \mathbb{N}$ without loss of generality.

To construct a suitable open cover $(\widetilde{U}_n)_{n \in \mathbb{N}}$, choose a sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets $K_n \subseteq M$ such that $\bigcup_{n \in \mathbb{N}} K_n = M$ and such that K_n is contained in the interior $(K_{n+1})^0$, for each $n \in \mathbb{N}$. Define $K_{-1} := K_0 := \emptyset$ for convenience of notation. Then $I_m := \{n \in \mathbb{N} \colon (K_m \setminus (K_{m-1})^0) \cap \overline{U_n} \neq \emptyset\}$ is a finite set, for each $m \in \mathbb{N}$, because also the sequence $(\overline{U_n})_{n \in \mathbb{N}}$ of the closures is locally finite.[10] Also $J_n := \{m \in \mathbb{N} \colon n \in I_m\}$ is a finite set for each $n \in \mathbb{N}$: indeed, there is $m_0 \in \mathbb{N}$ such that $\overline{U_n} \subseteq K_{m_0}$; then $\overline{U_n} \subseteq (K_m)^0$ for all $m \geq m_0 + 1$ and thus $\overline{U_n} \cap (K_m \setminus (K_{m-1})^0) = \emptyset$ for all $m \geq m_0 + 2$, entailing that $m \notin J_n$ for all $m \geq m_0 + 2$. For each $m \in \mathbb{N}$ and $n \in I_m$, the set $V_m := (K_{m+1})^0 \setminus K_{m-2}$ is an open neighbourhood of $\overline{U_n} \cap (K_m \setminus (K_{m-1})^0)$, which is contained in K_{m+1} and therefore relatively compact. We set $\widetilde{U}_n := \bigcup_{m \in J_n} V_m$; this is a relatively compact, open neighbourhood of $\overline{U_n}$. Given $n, m \in \mathbb{N}$, we have $K_m \cap \widetilde{U}_n = \bigcup_{m' \in J_n} (K_m \cap V_{m'})$, where $K_m \cap V_{m'} = \emptyset$ unless $m' \leq m + 2$. Let $m' \leq m + 2$. If $m' \in J_n$, then $n \in I_{m'}$. Thus $K_m \cap \widetilde{U}_n = \emptyset$ unless $n \in \bigcup_{m'=1}^{m+2} I_{m'}$, which is a finite set. It now readily follows that the open cover $(\widetilde{U}_n)_{n \in \mathbb{N}}$ of M is locally finite. In fact, given any $x \in M$ we find $m \in \mathbb{N}$ such that K_m^0 is an open neighbourhood of x. By the preceding, K_m (and hence K_m^0) only meets $\widetilde{U}_n$ for finitely many n. □

Cut-offs and partitions of unity on finite-dimensional real manifolds are standard tools. To enable unified proofs, we now discuss analogous concepts also over local fields.

Definition 8.6 Let $\mathbb{F}$ be a local field. A *$C^r_{\mathbb{F}}$-partition of unity* of a finite-dimensional $C^r_{\mathbb{F}}$-manifold M is a family $(h_i)_{i \in I}$ of continuous mappings $h_i \colon M \to \{0, 1\} \subseteq \mathbb{F}$, such that the open and closed sets $h_i^{-1}(\{1\})$ are mutually disjoint and cover M.

Note that, being locally constant, each h_i is actually $C^r_{\mathbb{F}}$.

Lemma 8.7 *Let M be a σ-compact $C^r_{\mathbb{F}}$-manifold over a local field $\mathbb{F}$, and $(U_i)_{i \in I}$ be an open cover of M. Then there exists a partition of unity $(h_i)_{i \in I}$ such that* $\operatorname{supp}(h_i) \subseteq U_i$.

[10] Every $x \in M$ has an open neighbourhood U such that $\{n \in \mathbb{N} \colon U_n \cap U \neq \emptyset\}$ is finite. The set U being open, we have $\{n \in \mathbb{N} \colon U_n \cap U \neq \emptyset\} = \{n \in \mathbb{N} \colon \overline{U_n} \cap U \neq \emptyset\}$.

Proof. The assertion is trivial if $\dim(M) = 0$. If $d := \dim(M) > 0$, by Lemma 8.3 we may assume that M is an open subset of $\mathbb{F}^d$. Let $(B_{r_j}(a_j))_{j\in J}$ and $i(j)$ for $j \in J$ be as in Lemma 8.4. The family of balls being locally finite, the open sets $V_i := \bigcup_{j\in J: i(j)=i} B_{r_j}(a_j) \subseteq U_i$ are also closed. For each $i \in I$, define $h_i\colon M \to \mathbb{F}$ via $h_i(x) := 1 \in \mathbb{F}$ if $x \in V_i$, $h_i(x) := 0$ otherwise. Then $(h_i)_{i\in I}$ is a partition of unity with the desired properties. □

Lemma 8.8 *Let $\mathbb{F}$ be a local field, M be a finite-dimensional $C^r_{\mathbb{F}}$-manifold, $K \subseteq M$ be compact, and $U \subseteq M$ be an open subset containing K. Then there exists a $C^r_{\mathbb{F}}$-function $h\colon M \to \{0,1\} \subseteq \mathbb{F}$ such that $h|_K = 1$ and $h|_{M\setminus U} = 0$.*

Proof. As each element $x \in K$ is contained in some open and compact ball $B_x \subseteq U$, exploiting the compactness of K we find finitely many open and compact balls $C_1, \ldots, C_n \subseteq U$ such that $K \subseteq \bigcup_{k=1}^n C_k =: W$. Then W is an open and closed neighbourhood of K such that $W \subseteq U$, and hence $h\colon M \to \mathbb{F}$, $h(x) := 1$ if $x \in W$, else $h(x) := 0$ is a function with the desired properties. □

Topologies on spaces of vector-valued test functions

For the remainder of this section, $\mathbb{F}$ denotes a locally compact, non-discrete topological field, and $\mathbb{K}$ a topological extension field of $\mathbb{F}$, whose topology arises from an absolute value $|.|\colon \mathbb{K} \to [0,\infty[$. We let $r \in \mathbb{N}_0 \cup \{\infty\}$.

Definition 8.9 Given a paracompact $C^r_{\mathbb{F}}$-manifold M, modeled on a finite-dimensional $\mathbb{F}$-vector space Z, and a (Hausdorff, not necessarily locally convex) topological $\mathbb{K}$-vector space E, we let

$$C^r_c(M,E) := \{\gamma \in C^r(M,E)\colon\ \operatorname{supp}(\gamma) \text{ is compact}\,\}$$

be the set of compactly supported E-valued $C^r_{\mathbb{F}}$-functions on M. Then $C^r_c(M,E)$ is a $\mathbb{K}$-vector subspace of $C^r(M,E)$, and $C^r_c(M,E) = \bigcup_{K\in\mathcal{K}(M)} C^r_K(M,E)$, where $\mathcal{K}(M)$ denotes the set of all compact subsets of M. In the following, we consider three vector topologies on $C^r_c(M,E)$:

(a) We write $C^r_c(M,E)_{\text{tvs}}$ for $C^r_c(M,E)$, equipped with the finest (a priori not necessarily Hausdorff) vector topology making the inclusion maps $\lambda_K\colon C^r_K(M,E) \to C^r_c(M,E)$ continuous for each compact subset $K \subseteq M$. Thus $C^r_c(M,E)_{\text{tvs}} = \varinjlim C^r_K(M,E)$ in the category of not necessarily Hausdorff topological $\mathbb{K}$-vector spaces and continuous $\mathbb{K}$-linear maps.

(b) If E is locally convex, we write $C^r_c(M,E)_{\text{lcx}}$ for $C^r_c(M,E)$, equipped with the finest (a priori not necessarily Hausdorff) locally convex vector topology making the inclusion maps $\lambda_K\colon C^r_K(M,E) \to C^r_c(M,E)$ continuous for each compact subset $K \subseteq M$. Thus $C^r_c(M,E)_{\text{lcx}} = \varinjlim C^r_K(M,E)$ in the category of not necessarily Hausdorff, locally convex topological $\mathbb{K}$-vector spaces and continuous $\mathbb{K}$-linear maps.

(c) Given a locally finite cover $\mathcal{U} = (U_i)_{i \in I}$ of M by relatively compact, open subsets[11] $U_i \subseteq M$, we let $\rho_i \colon C^r_c(M,E) \to C^r(U_i,E)$ be the restriction map for $i \in I$ and define

$$\rho_{\mathcal{U}} \colon C^r_c(M,E) \to \bigoplus_{i \in I} C^r(U_i,E), \quad \rho_{\mathcal{U}}(\gamma) := (\rho_i(\gamma))_{i \in I} = (\gamma|_{U_i})_{i \in I}\,.$$

We write $C^r_c(M,E)_{\text{box}}$ for $C^r_c(M,E)$, equipped with the topology $\mathcal{O}_{\mathcal{U}}$ induced by $\rho_{\mathcal{U}}$, where the direct sum is endowed with the box topology.

Lemma 8.10 *In the situation of Definition* 8.9 (c), *assume that both* $\mathcal{U} = (U_i)_{i \in I}$ *and* $\mathcal{V} = (V_j)_{j \in J}$ *are locally finite covers of* M *by relatively compact open subsets. Then* $\mathcal{O}_{\mathcal{U}} = \mathcal{O}_{\mathcal{V}}$*. In other words, the box topology on* $C^r_c(M,E)$ *is independent of the choice of* $\mathcal{U}$*.*

Proof. The topologies $\mathcal{O}_{\mathcal{U}}$ and $\mathcal{O}_{\mathcal{V}}$ are induced by $\rho_{\mathcal{U}} : C^r_c(M,E) \to \bigoplus_{i \in I} C^r(U_i,E)$, $\rho_{\mathcal{U}}(\gamma) := (\gamma|_{U_i})_{i \in I}$ and $\rho_{\mathcal{V}} \colon C^r_c(M,E) \to \bigoplus_{j \in J} C^r(V_j,E)$, $\rho_{\mathcal{V}}(\gamma) := (\gamma|_{V_j})_{j \in J}$, respectively.

8.11 Given $i \in I$, the set $J_i := \{j \in J \colon U_i \cap V_j \neq \emptyset\}$ is finite, as U_i is relatively compact and $\mathcal{V}$ is a locally finite cover. By Lemma 4.12, the topology on $C^r(U_i,E)$ is initial with respect to the family $(\mu_{i,j})_{j \in J_i}$ of restriction maps

$$\mu_{i,j} \colon C^r(U_i,E) \to C^r(U_i \cap V_j, E), \quad \mu_{i,j}(\gamma) := \gamma|_{U_i \cap V_j}\,.$$

Likewise, the set $I_j := \{i \in I : U_i \cap V_j \neq \emptyset\}$ is finite for each $j \in J$, and the topology on $C^r(V_j,E)$ is initial with respect to the family $(\nu_{j,i})_{i \in I_j}$ of restriction mappings $\nu_{j,i} \colon C^r(V_j,E) \to C^r(U_i \cap V_j, E)$.

8.12 Let $P_{i,j}$ be an open 0-neighbourhood of $C^r(U_i \cap V_j, E)$, for any $i \in I$, $j \in J_i$. Then $P_i := \bigcap_{j \in J_i} \mu_{i,j}^{-1}(P_{i,j})$ is an open 0-neighbourhood in $C^r(U_i,E)$ for each $i \in I$ and thus

$$P := \bigoplus_{i \in I} P_i$$

is an open 0-neighbourhood in $\bigoplus_{i \in I} C^r(U_i,E)$. It is clear from the preceding that the set $\mathcal{B}$ of such open 0-neighbourhoods P is a basis for the filter of 0-neighbourhoods of $\bigoplus_{i \in I} C^r(U_i,E)$, and hence $\{\rho_{\mathcal{U}}^{-1}(P) \colon P \in \mathcal{B}\}$ is a basis for the filter of 0-neighbourhoods of $(C^r_c(M,E), \mathcal{O}_{\mathcal{U}})$.

To see that $\mathcal{O}_{\mathcal{U}} = \mathcal{O}_{\mathcal{V}}$, it suffices to show that $\mathcal{O}_{\mathcal{U}} \subseteq \mathcal{O}_{\mathcal{V}}$ (as we can interchange $\mathcal{U}$ and $\mathcal{V}$). Since both $\mathcal{O}_{\mathcal{U}}$ and $\mathcal{O}_{\mathcal{V}}$ are vector topologies, we only need to show that $W \in \mathcal{O}_{\mathcal{V}}$ for W ranging through a suitable basis of open 0-neighbourhoods of $(C^r_c(M,E), \mathcal{O}_{\mathcal{U}})$. It therefore suffices to consider $W := \rho_{\mathcal{U}}^{-1}(P)$ for $P \in \mathcal{B}$ as in **8.12**. Set $Q_j := \bigcap_{i \in I_j} \nu_{j,i}^{-1}(P_{i,j})$ for $j \in J$;

[11] Such a cover always exists because M is locally compact and paracompact.

then $Q := \bigoplus_{j\in J} Q_j$ is an open 0-neighbourhood in $\bigoplus_{j\in J} C^r(V_j, E)$. For $\gamma \in C^r_c(M,E)$, we have

$$\begin{aligned}
\gamma \in W = \rho_{\mathcal{U}}^{-1}(P) \quad &\Leftrightarrow \quad \rho_{\mathcal{U}}(\gamma) \in P \quad \Leftrightarrow \quad (\forall i \in I)\ \gamma|_{U_i} \in P_i \\
&\Leftrightarrow \quad (\forall i \in I)\,(\forall j \in J_i) \quad \gamma|_{U_i \cap V_j} \in P_{i,j} \\
&\Leftrightarrow \quad (\forall j \in J)\,(\forall i \in I_j) \quad \gamma|_{U_i \cap V_j} \in P_{i,j} \\
&\Leftrightarrow \quad (\forall j \in J)\ \gamma|_{V_j} \in Q_j \quad \Leftrightarrow \quad \rho_{\mathcal{V}}(\gamma) \in Q \quad \Leftrightarrow \quad \gamma \in \rho_{\mathcal{V}}^{-1}(Q)\,.
\end{aligned}$$

Thus $W = \rho_{\mathcal{V}}^{-1}(Q) \in \mathcal{O}_{\mathcal{V}}$, which completes the proof. □

Proposition 8.13 *Let M be a paracompact $C^r_{\mathbb{F}}$-manifold, modeled on a finite-dimensional $\mathbb{F}$-vector space Z, and E be a topological $\mathbb{K}$-vector space. Then the following holds:*

(a) *The box topology on $C^r_c(M,E)_{\mathrm{box}}$ is Hausdorff. For every locally finite cover $\mathcal{U} = (U_i)_{i\in I}$ of M by relatively compact, open subsets $U_i \subseteq M$, the map*

$$\rho_{\mathcal{U}} \colon C^r_c(M,E)_{\mathrm{box}} \to \bigoplus_{i\in I} C^r(U_i, E), \quad \rho_{\mathcal{U}}(\gamma) := (\gamma|_{U_i})_{i\in I}$$

has closed image, and $\rho_{\mathcal{U}}|^{\mathrm{im}\,\rho_{\mathcal{U}}}$ is an isomorphism of topological vector spaces. The inclusion map $C^r_c(M,E)_{\mathrm{box}} \to C^r(M,E)$ is continuous. If E is locally convex, then $C^r_c(M,E)_{\mathrm{box}}$ is locally convex.

(b) *The inclusion map $\lambda_K : C^r_K(M,E) \to C^r_c(M,E)_{\mathrm{box}}$ is continuous and induces the given topology on $C^r_K(M,E)$, for each compact subset $K \subseteq M$.*

(c) *The map $\Phi \colon C^r_c(M,E)_{\mathrm{tvs}} \to C^r_c(M,E)_{\mathrm{box}}$, $\Phi(\gamma) := \gamma$ is continuous. Thus $C^r_c(M,E)_{\mathrm{tvs}}$ is Hausdorff and induces the given topology on each $C^r_K(M,E)$. If $\mathbb{F} \neq \mathbb{C}$ and M is σ-compact, then Φ is an isomorphism of topological $\mathbb{K}$-vector spaces.*

(d) *If E is locally convex, then $\Psi \colon C^r_c(M,E)_{\mathrm{lcx}} \to C^r_c(M,E)_{\mathrm{box}}$, $\Psi(\gamma) := \gamma$ is continuous. Hence $C^r_c(M,E)_{\mathrm{lcx}}$ is Hausdorff and induces the given topology on each $C^r_K(M,E)$. If $\mathbb{F} \neq \mathbb{C}$ and M is σ-compact, then Ψ is an isomorphism of topological $\mathbb{K}$-vector spaces.*

(e) *If $\mathbb{F}$ is a local field and $\mathcal{U} = (U_i)_{i\in I}$ is a cover of M by mutually disjoint, compact open sets (cf. Lemma 8.3 (b)), then*

$$\rho_{\mathcal{U}} \colon C^r_c(M,E)_{\mathrm{box}} \to \bigoplus_{i\in I} C^r(U_i, E), \quad \rho_{\mathcal{U}}(\gamma) := (\gamma|_{U_i})_{i\in I}$$

is an isomorphism of topological vector spaces onto the direct sum, equipped with the box topology.

(f) *If $\mathbb{F}$ is a local field and E is locally convex, then Ψ is an isomorphism of topological vector spaces, i.e., $C^r_c(M,E)_{\mathrm{lcx}} = C^r_c(M,E)_{\mathrm{box}}$.*

In particular, $C^r_c(M,E)_{\mathrm{box}} = C^r_c(M,E)_{\mathrm{tvs}} = C^r_c(M,E)_{\mathrm{lcx}}$ *if* $\mathbb{F} \neq \mathbb{C}$ *and* M *is* σ*-compact.*

Proof. (a) Let $\mathcal{U} = (U_i)_{i\in I}$ be a locally finite cover of M by relatively compact, open sets. The box topology on $\bigoplus_{i\in I} C^r(U_i,E)$ being Hausdorff and $\rho_{\mathcal{U}}\colon C^r_c(M,E) \to \bigoplus_{i\in I} C^r(U_i,E)$ being injective, the topology $\mathcal{O}_{\mathrm{box}}$ induced by $\rho_{\mathcal{U}}$ on $C^r_c(M,E)$ is Hausdorff and $\rho_{\mathcal{U}}|^{\mathrm{im}\,\rho_{\mathcal{U}}}$ is an isomorphism of topological vector spaces. The box topology on $S := \bigoplus_{i\in I} C^r(U_i,E)$ is properly finer than the topology induced by the product $P := \prod_{i\in I} C^r(U_i,E)$. The map $\tau : C^r(M,E) \to P$, $\tau(\gamma) := (\gamma|_{U_i})_{i\in I}$ is a topological embedding with closed image, by Lemma 4.12. This entails, firstly, that the inclusion map $C^r_c(M,E)_{\mathrm{box}} \to C^r(M,E)$ is continuous. Secondly, it entails that $\mathrm{im}(\tau)\cap S$ is closed in S. Note that $\tau(\gamma) \in S$ implies that $\mathrm{supp}(\gamma)$ is compact, i.e., $\gamma \in C^r_c(M,E)$. Thus $\mathrm{im}(\rho_{\mathcal{U}}) = \mathrm{im}(\tau)\cap S$ is closed in S. If E is locally convex, then each of the spaces $C^r(U_i,E)$ is locally convex (Proposition 4.19 (b)), whence so is the direct sum $\bigoplus_{i\in I} C^r(U_i,E)$ (see **6.4**) and hence so is $\mathcal{O}_{\mathrm{box}}$.

(b) Let $K \subseteq M$ be compact. For $\mathcal{U} = (U_i)_{i\in I}$ as before, there exists a finite subset $J \subseteq I$ such that $U_i \cap K = \emptyset$ for all $i \in I \setminus J$. Thus $K \subseteq \bigcup_{i\in J} U_i =: W$. Then the composition $f\colon C^r_K(M,E) \to \bigoplus_{i\in I} C^r(U_i,E)$ of the maps

$$C^r_K(M,E) \xrightarrow{\cong} C^r_K(W,E) \hookrightarrow C^r(W,E) \hookrightarrow \prod_{i\in J} C^r(U_i,E) \hookrightarrow \bigoplus_{i\in I} C^r(U_i,E)$$

is a topological embedding, where the first map and the coordinate functions of the second map are the respective restriction maps (see Lemma 4.24 and Lemma 4.12), and the last map is inclusion (see **6.3**). Since $f = \rho_{\mathcal{U}} \circ \lambda_K$, where $\rho_{\mathcal{U}}$ is a topological embedding, we deduce that also λ_K is a topological embedding.

(c) Since $\lambda_K : C^r_K(M,E) \to C^r_c(M,E)_{\mathrm{box}}$ is continuous for each $K \in \mathcal{K}(M)$, our definition of $C^r_c(M,E)_{\mathrm{tvs}}$ shows that the topology on $C^r_c(M,E)_{\mathrm{tvs}}$ is finer than the one on $C^r_c(M,E)_{\mathrm{box}}$, and thus Φ is continuous. Since $\lambda_K\colon C^r_K(M,E) \to C^r_c(M,E)_{\mathrm{tvs}}$ is continuous as a map into $C^r_c(M,E)_{\mathrm{tvs}}$ and $\Phi\circ\lambda_K\colon C^r_K(M,E) \to C^r_c(M,E)_{\mathrm{box}}$ is a topological embedding by (b), also $\lambda_K\colon C^r_K(M,E) \to C^r_c(M,E)_{\mathrm{tvs}}$ is a topological embedding.

We now assume that M is σ-compact, and we assume that $\mathbb{F}$ is not isomorphic to $\mathbb{C}$ as a topological field; then $\mathbb{F}$ is a local field or $\mathbb{F} \cong \mathbb{R}$ (see [78]). We have to show that $C^r_c(M,E)_{\mathrm{box}} = \varinjlim C^r_K(M,E)$ in the category of topological vector spaces, with limit maps $\lambda_K : C^r_K(M,E) \to C^r_c(M,E)_{\mathrm{box}}$. We already know from (b) that each λ_K is continuous; thus $(C^r_c(M,E)_{\mathrm{box}}, (\lambda_K)_{K\in\mathcal{K}(M)})$ is a cone in the category of topological $\mathbb{K}$-vector spaces and continuous $\mathbb{K}$-linear maps. To see that it is a direct limit cone, suppose that $(f_K)_{K\in\mathcal{K}(M)}$ is a family of continuous linear maps $f_K\colon C^r_K(M,E) \to F$ into a topological $\mathbb{K}$-vector space F such that $f_L|_{C^r_K(M,E)} = f_K$ whenever $K \subseteq L$. Then $f\colon C^r_c(M,E)_{\mathrm{box}} \to F$, $f(\gamma) := f_K(\gamma)$ if $\mathrm{supp}(\gamma) \subseteq K$ is well-defined and is the unique linear map $C^r_c(M,E)_{\mathrm{box}} \to F$ such that $f\circ\lambda_K = f_K$ for each K. To establish the desired direct limit property, it only remains to show that f is continuous.

Let $\mathcal{U} = (U_i)_{i\in I}$ be as before; M being σ-compact, we may assume that I is countable. We pick a $C^r_{\mathbb{F}}$-partition of unity $(h_i)_{i\in I}$ of M such that $\mathrm{supp}(h_i) \subseteq U_i$ for each $i \in I$

(see Lemma 8.7 when $\mathbb{F}$ is a local field; the real case is standard). Since U_i is relatively compact, $K_i := \operatorname{supp}(h_i) \subseteq U_i$ is compact. Let $e_i : C^r_{K_i}(U_i, E) \to C^r_{K_i}(M, E)$ be the isomorphism of topological vector spaces which extends functions by 0 (cf. Lemma 4.24). Then $g_i : C^r(U_i, E) \to F$, $g_i := f_{K_i} \circ e_i \circ \mu_{h_i}$ is a continuous linear mapping, where $\mu_{h_i} : C^r(U_i, E) \to C^r_{K_i}(U_i, E)$ is the multiplication operator defined via $\mu_{h_i}(\gamma) := h_i|_{U_i} \cdot \gamma$, which is continuous linear (in view of Lemma 1.15 and Lemma 4.12, applied with a cover of coordinate neighbourhoods, this assertion can be reduced to Lemma 4.5). By the universal property of the countable direct sum $S := \bigoplus_{i \in I} C^r(U_i, E)$ (see **6.5**), the linear map

$$g\colon S \to F, \quad (\gamma_i)_{i \in I} \mapsto \sum_{i \in I} g_i(\gamma_i)$$

(where $\gamma_i \in C^r(U_i, E)$) is continuous, because so is each g_i. Given $\gamma \in C^r_c(M, E)$, we calculate $g_i(\gamma|_{U_i}) = f(e_i(h_i|_{U_i} \cdot \gamma|_{U_i})) = f(e_i((h_i \cdot \gamma)|_{U_i})) = f(h_i \cdot \gamma)$, whence $g(\rho_{\mathcal{U}}(\gamma)) = \sum_{i \in I} g_i(\gamma|_{U_i}) = \sum_{i \in I} f(h_i \cdot \gamma) = f(\sum_{i \in I} h_i \cdot \gamma) = f(\gamma)$. Thus $g \circ \rho_{\mathcal{U}} = f$, and so f is continuous on $C^r_c(M, E)_{\mathrm{box}}$, as required; the direct limit property is fully established.

(d) Since $C^r_c(M, E)_{\mathrm{box}}$ is locally convex for locally convex E (see (a)), we can repeat the proof of (c), except that topological vector spaces have to be replaced with locally convex spaces.

(e) By the definition of the box topology, $\rho_{\mathcal{U}}$ is a topological embedding. Each of the sets U_i being compact and open, given $(\gamma_i)_{i \in I} \in \bigoplus_{i \in I} C^r(U_i, E)$ the map $\gamma : M \to E$ defined via $\gamma(x) := \gamma_i(x)$ for $x \in U_i$ is $C^r_{\mathbb{F}}$ and compactly supported. Thus $\gamma \in C^r_c(M, E)$, and $\rho_{\mathcal{U}}(\gamma) = (\gamma_i)_{i \in I}$ by definition of γ. Thus $\rho_{\mathcal{U}}$ is also surjective, and thus $\rho_{\mathcal{U}}$ is an isomorphism of topological vector spaces.

(f) Assume that $f : C^r_c(M, E)_{\mathrm{box}} \to F$ is a linear map into a locally convex topological $\mathbb{K}$-vector space F such that $f \circ \lambda_K$ is continuous for each $K \in \mathcal{K}(M)$. Let $\mathcal{U} = (U_i)_{i \in I}$ be as in (e). Then $f|_{C^r_{U_i}(M,E)}$ is continuous in particular for each $i \in I$, and hence so is $g_i := f \circ e_i : C^r(U_i, E) \to F$, where $e_i : C^r(U_i, E) \to C^r_{U_i}(M, E)$ is the isomorphism of topological vector spaces obtained as the inverse of the restriction map $C^r_{U_i}(M, E) \to C^r(U_i, E)$ (see Lemma 4.24). Since the box topology makes $\bigoplus_{i \in I} C^r(U_i, E)$ the category-theoretical locally convex direct sum in the present situation (see Remark 6.7), the map $g\colon \bigoplus_{i \in I} C^r(U_i, E) \to F$, $g((\gamma_i)_{i \in I}) := \sum_{i \in I} g_i(\gamma_i)$ is continuous linear. Hence also $f = g \circ \rho_{\mathcal{U}}$ is continuous. □

Convention 8.14 Throughout the following, spaces of vector-valued test functions will always be equipped with the box topology, and we abbreviate $C^r_c(M, E) := C^r_c(M, E)_{\mathrm{box}}$.

Remark 8.15 If $\mathbb{F} = \mathbb{K} = \mathbb{R}$, M is σ-compact and E is locally convex, then the box topology on $C^r_c(M, E)$ coincides with the locally convex topology traditionally considered on this space of test functions, by Proposition 4.19 (d) and Proposition 8.13 (d).

Remark 8.16 As we shall mainly need spaces of test functions with values in locally convex spaces over local fields in the following (for example, in our discussion of diffeomorphism groups), we have chosen to work with the box topology, which is the appropriate topology on $C^r_c(M,E)$ in this case, even for non-σ-compact M (see Proposition 8.13 (f)). If M is a non-σ-compact, paracompact finite-dimensional manifold over $\mathbb{R}$ and E a real locally convex space, it is certainly more natural to work with the (finer) locally convex direct limit topology on $C^r_c(M,E)$, and study differentiability properties of mappings between spaces of test functions topologized in this way. We did not find it advantageous (nor necessary) to discuss this situation in parallel here; the interested reader can find a separate discussion in [32] and [33] (cf. also [57]).

Remark 8.17 Spaces of compactly supported sections in vector bundles can be treated much in the same way as spaces of test functions. However, the only vector bundles we shall really have to work with in our Lie group constructions (of diffeomorphism groups) are the tangent bundles of paracompact finite-dimensional smooth manifolds M over local fields $\mathbb{K}$. Since any vector bundle over such a manifold M is trivial (as a consequence of Lemma 8.3 (b) and Lemma 8.4), it is not necessary for our purposes to introduce the additional machinery required to discuss spaces of sections and vector bundles, and so we decided to defer their discussion to an appendix (Appendix F). The only facts we shall really use are the following: 1. For each paracompact, finite-dimensional $C^r_{\mathbb{K}}$-manifold M and disjoint cover $(B_i)_{i\in I}$ by open and compact balls, the map $C^r_c(M,TM)\to\bigoplus_{i\in I}C^r(B_i,TB_i)$, $\sigma\mapsto(\sigma|_{B_i})_{i\in I}$ is an isomorphism of topological vector spaces (Proposition F.19 (e)). 2. If $\kappa\colon M\to B$ is a $C^r_{\mathbb{K}}$-diffeomorphism from a d-dimensional $C^r_{\mathbb{K}}$-manifold M onto a metric ball $B\subseteq\mathbb{K}^d$, then $C^r(M,TM)\to C^r(B,\mathbb{K}^d)$, $\sigma\mapsto(x\mapsto(d\kappa_i)\circ\sigma\circ\kappa^{-1})$ is an isomorphism of topological $\mathbb{K}$-vector spaces (cf. Lemma F.9 and Lemma 4.9).

Patched topological vector spaces and patched mappings

To formalize the situation encountered in Proposition 8.13 (a), we now introduce the notion of a "patched" topological vector space. Roughly speaking, this is a topological vector space, together with an embedding into a direct sum. We then discuss differentiability properties of mappings between patched topological vector spaces. The general results obtained here shall allow us to transfer our discussion of pushforwards to the case of test functions (Proposition 8.22). We shall also derive certain very convenient criteria ensuring differentiability properties for mappings between spaces of test functions (Section 10).

For analogous discussions (and applications) of "patched locally convex spaces" based on locally convex direct sums, we refer to [32] and [33].

Definition 8.18 A *patched topological vector space* over a valued field $(\mathbb{K},|.|)$ is a pair $(E,(p_i)_{i\in I})$, where E is a topological $\mathbb{K}$-vector space and $(p_i)_{i\in I}$ a family of continuous linear maps $p_i\colon E\to E_i$ to certain topological vector spaces E_i, such that

(a) For each $x\in E$, the set $\{i\in I\colon p_i(x)\neq 0\}$ is finite;

(b) The linear map

$$p\colon E \to \bigoplus_{i\in I} E_i\,, \quad x\mapsto (p_i(x))_{i\in I}=\sum_{i\in I} p_i(x)$$

from E to the direct sum $\bigoplus_{i\in I} E_i$ (equipped with the box topology) is a topological embedding;

(c) The image $p(E)$ is sequentially closed in $\bigoplus_{i\in I} E_i$.

The mappings $p_i\colon E\to E_i$ are called *patches*, and the family $(p_i)_{i\in I}$ is called a *patchwork*.

We retain the notation introduced earlier in this section.

Example 8.19 Let M be a paracompact, finite-dimensional $C^r_{\mathbb{F}}$-manifold over $\mathbb{F}$ (where $r\in\mathbb{N}_0\cup\{\infty\}$), and E be a topological $\mathbb{K}$-vector space. Let $(U_i)_{i\in I}$ be a locally finite open cover of M by relatively compact, open subsets $U_i\subseteq M$ and $\rho_i\colon C^r_c(M,E)\to C^r(U_i,E)$, $\rho_i(\gamma):=\gamma|_{U_i}$ be the restriction map for $i\in I$. Then

$$(C^r_c(M,E),(\rho_i)_{i\in I})$$

is a patched topological vector space, by Proposition 8.13 (a).

We now discuss mappings between open subsets of patched topological vector spaces.

Definition 8.20 Let $(E,(p_i)_{i\in I})$ and $(F,(q_i)_{i\in I})$ be patched topological $\mathbb{K}$-vector spaces over the same index set I. Let $p\colon E\to\bigoplus_{i\in I}E_i$ and $q\colon F\to\bigoplus_{i\in I}F_i$ be the canonical embeddings.

(a) A map $f\colon U\to F$, defined on an open subset U of E, is called a *patched mapping* if there exists a family $(f_i)_{i\in I}$ of mappings $f_i\colon U_i\to F_i$ on certain open neighbourhoods U_i of $p_i(U)$ in E_i, which is *compatible with* f in the following sense: we have $0\in U_i$ and $f_i(0)=0$ for all but finitely many i, and $q_i(f(x))=f_i(p_i(x))$ for all $i\in I$, *i.e.*, $q\circ f=(\oplus f_i)\circ p|_U^{\oplus U_i}$.

(b) Given $k\in\mathbb{N}_0\cup\{\infty\}$, we say that a patched mapping $f\colon U\to F$ as before is *of class $C^k_{\mathbb{K}}$ on the patches* if all of the mappings f_i in (a) can be chosen of class $C^k_{\mathbb{K}}$.

Proposition 8.21 *Let $(E,(p_i)_{i\in I})$ and $(F,(q_i)_{i\in I})$ be patched topological $\mathbb{K}$-vector spaces over the same index set I. Assume that $f\colon U\to F$ is a patched mapping from an open subset $U\subseteq E$ to F. If f is of class $C^k_{\mathbb{K}}$ on the patches, then f is of class $C^k_{\mathbb{K}}$.*

Proof. Let $p_i\colon E\to E_i$ and $q_i\colon F\to F_i$ be the patches of E, resp., F. If f is of class $C^k_{\mathbb{K}}$ on the patches, then there exists a family $(f_i)_{i\in I}$ of $C^k_{\mathbb{K}}$-maps $f_i\colon U_i\to F_i$, which is compatible with f. By Proposition 6.9, the map $g:=\oplus_{i\in I}f_i:\bigoplus_{i\in I}U_i\to\bigoplus_{i\in I}F_i$, $g(\sum_{i\in I}u_i):=\sum_{i\in I}f_i(u_i)$ is of class $C^k_{\mathbb{K}}$. The linear map $p:E\to\bigoplus_{i\in I}E_i$, $p(x):=\sum_{i\in I}p_i(x)$ being continuous, the composition $g\circ p|_U^{\oplus U_i}$ is $C^k_{\mathbb{K}}$. But $g\circ p|_U^{\oplus U_i}=q\circ f$, where $q\colon F\to\bigoplus_{i\in I}F_i$,

$q(y) := \sum_{i\in I} q_i(y)$, since $g(p(x)) = \sum_{i\in I} f_i(p_i(x)) = \sum_{i\in I} q_i(f(x)) = q(f(x))$ for all $x \in U$. By the preceding, the map $q\circ f$ is of class $C^k_{\mathbb{K}}$. Its image being contained in the sequentially closed vector subspace $Q := \operatorname{im} q$ of $\bigoplus_{i\in I} F_i$, Lemma 1.15 shows that also the co-restriction $(q\circ f)|^Q$ is of class $C^k_{\mathbb{K}}$. As $q|^Q$ is an isomorphism of topological vector spaces (by the axioms of a patched topological vector space), we see that also $f = (q|^Q)^{-1} \circ (q \circ f)|^Q$ is $C^k_{\mathbb{K}}$. □

Example: Pushforwards of compactly supported functions

We now establish an analogue of Proposition 4.20 for pushforwards between spaces of test functions. The idea is to use the technique of patched topological vector spaces to reduce the assertion to Proposition 4.23 (a). A further generalization to mappings between spaces of compactly supported sections in vector bundles is provided in Appendix F (the "Ω-Lemma with parameters"). In the real locally convex case, stronger and much more refined results are available: see [32].

Proposition 8.22 *Let E, F and $\widetilde{Z}$ be topological $\mathbb{K}$-vector spaces, $U \subseteq E$ an open zero-neighbourhood, $r, k \in \mathbb{N}_0 \cup \{\infty\}$, $\widetilde{M}$ be a $\mathbb{K}$-manifold of class $C^{r+k}_{\mathbb{K}}$ modeled on $\widetilde{Z}$, and $\tilde{f}\colon \widetilde{M} \times U \to F$ be a mapping of class $C^{r+k}_{\mathbb{K}}$. Let M be a paracompact, finite-dimensional $\mathbb{F}$-manifold of class $C^r_{\mathbb{F}}$. Given a mapping $\sigma\colon M \to \widetilde{M}$ of class $C^r_{\mathbb{F}}$, we define $f := \tilde{f} \circ (\sigma \times \mathrm{id}_U)\colon M \times U \to F$. We assume that there exists a compact subset $K \subseteq M$ such that $f(x, 0) = 0$ for all $x \in M \setminus K$. Then $C^r_c(M, U) := \{\gamma \in C^r_c(M, E)\colon \gamma(M) \subseteq U\}$ is an open subset of $C^r_c(M, E)$, equipped with the box topology, and*

$$f_*\colon C^r_c(M, U) \to C^r_c(M, F), \quad f_*(\gamma)(x) := f(x, \gamma(x))$$

is a mapping of class $C^k_{\mathbb{K}}$.

Proof. There exist locally finite covers $\mathcal{V} := (V_i)_{i\in I}$ and $\mathcal{U} := (U_i)_{i\in I}$ of M by relatively compact, open sets V_i (resp., U_i), such that $K_i := \overline{V_i} \subseteq U_i$ for all $i \in I$ (cf. Lemma 8.5).

To see that $C^r_c(M, U)$ is open in $C^r_c(M, E)$, note that $\lfloor K_i, U\rfloor_r \subseteq C^r(U_i, E)$ is an open 0-neighbourhood, and thus $Q := \bigoplus_{i\in I} \lfloor K_i, U\rfloor_r$ is an open 0-neighbourhood in $\bigoplus_{i\in I} C^r(U_i, E)$. By definition of the box topology on $C^r(M, E)$, the map $\rho_{\mathcal{U}}\colon C^r_c(M, E) \to \bigoplus_{i\in I} C^r(U_i, E)$, $\rho_{\mathcal{U}}(\eta) := (\eta|_{U_i})_{i\in I}$ is continuous. Hence $\rho_{\mathcal{U}}^{-1}(Q)$ is an open 0-neighbourhood in $C^r_c(M, E)$. Since

$$\rho_{\mathcal{U}}^{-1}(Q) \;=\; \{\, \gamma \in C^r_c(M, E)\colon\ (\forall i \in I)\ \gamma(K_i) \subseteq U \,\}\,,$$

where $M = \bigcup_{i\in I} K_i$, we have $\rho_{\mathcal{U}}^{-1}(Q) = C^r_c(M, U)$. Hence $C^r_c(M, U)$ is an open 0-neighbourhood, as required.

To see that f_* is $C^k_{\mathbb{K}}$, we shall exploit that $(C^r_c(M, E), (\rho_i)_{i\in I})$ and $(C^r_c(M, F), (\tau_i)_{i\in I})$ are patched topological vector spaces, with the patches

$$\rho_i\colon C^r_c(M, E) \to C^r_c(U_i, E), \quad \rho_i(\gamma) := \gamma|_{U_i} \quad \text{and}$$

$$\tau_i \colon C^r_c(M,F) \to C^r_c(V_i,F), \quad \tau_i(\gamma) := \gamma|_{V_i}\,,$$

respectively (see Example 8.19). The map $f_i := f|_{U_i \times U} \colon U_i \times U \to F$ being of the form $\widetilde{f} \circ (\sigma|_{U_i} \times \mathrm{id}_U)$, the pushforward

$$(f_i)_* \colon \lfloor K_i, U \rfloor_r \to C^r(V_i, F)$$

is $C^r_{\mathbb{K}}$ on the open subset $\lfloor K_i, U\rfloor_r \subseteq C^r(U_i, E)$, by Proposition 4.23 (a) (applied with a singleton parameter set P). Apparently $\rho_i(C^r_c(M,U)) \subseteq \lfloor K_i, U\rfloor_r$ for each $i \in I$, and the family of mappings $((f_i)_*)_{i\in I}$ is compatible with f_*, i.e., $\tau_i \circ f_* = (f_i)_* \circ \rho_i$ for all $i \in I$. Thus f_* is a patched mapping. Every $(f_i)_*$ being $C^k_{\mathbb{K}}$, the map f_* is $C^k_{\mathbb{K}}$ on the patches and hence $C^k_{\mathbb{K}}$, by Proposition 8.21. □

Corollary 8.23 *Let E and F be topological $\mathbb{K}$-vector spaces and $f \colon U \to F$ be a mapping of class $C^{r+k}_{\mathbb{K}}$, defined on an open zero-neighbourhood $U \subseteq E$, such that $f(0) = 0$. Let M be a paracompact, finite-dimensional $\mathbb{F}$-manifold of class $C^r_{\mathbb{F}}$. Then*

$$C^r_c(M,f) \colon C^r_c(M,U) \to C^r_c(M,F), \quad \gamma \mapsto f \circ \gamma$$

is a mapping of class $C^k_{\mathbb{K}}$.

Proof. Let $\widetilde{M} := \{0\}$ be a singleton smooth $\mathbb{K}$-manifold, and $\sigma \colon M \to \widetilde{M}$, $x \mapsto 0$, which apparently is a $C^r_{\mathbb{F}}$-map. Then $\tilde{g} \colon \widetilde{M} \times U \to F$, $\tilde{g}(0,u) := f(u)$ is a mapping of class $C^{r+k}_{\mathbb{K}}$, and $C^r_K(M,f) = g_*$ for $g := \tilde{g} \circ (\sigma \times \mathrm{id}_U)$. By Proposition 8.22, g_* is $C^k_{\mathbb{K}}$. □

9 Test function groups and algebras of test functions

As in Section 8, let $\mathbb{F}$ be a locally compact topological field and $\mathbb{K}$ be a valued field which is a topological extension field of $\mathbb{F}$. Let $r \in \mathbb{N}_0 \cup \{\infty\}$. In view of Proposition 8.22 and Corollary 8.23, we can re-use the arguments from Section 5 to obtain the following:

Proposition 9.1 *Let M be a paracompact, finite-dimensional $\mathbb{F}$-manifold of class $C^r_{\mathbb{F}}$.*

(a) *If A is a topological $\mathbb{K}$-algebra, then also $C^r_c(M,A)$ is a topological $\mathbb{K}$-algebra (using pointwise multiplication).*

(b) *If A is an associative topological $\mathbb{K}$-algebra and E a topological A-module, then $C^r_c(M,E)$ is a topological $C^r_c(M,A)$-module.*

(c) *If G is a $\mathbb{K}$-Lie group modeled on a topological $\mathbb{K}$-vector space E, then*

$$C^r_c(M,G) := \{\gamma \in C^r(M,G) \colon \overline{\gamma^{-1}(G \setminus \{1\})} \text{ is compact}\}$$

can be given a $C^\infty_{\mathbb{K}}$-manifold structure modeled on the topological $\mathbb{K}$-vector space $C^r_c(M,E)$ in one and only one way, such that $C^r_c(M,G)$ becomes a $\mathbb{K}$-Lie group and such that $C^r_c(M,U_\phi) := C^r_c(M,G) \cap (U_\phi)^M$ is open in $C^r_c(M,G)$ and

$$C^r_c(M,\phi)\colon C^r_c(M,U_\phi) \to C^r_c(M,V_\phi), \quad \gamma \mapsto \phi \circ \gamma$$

is a $C^\infty_{\mathbb{K}}$-diffeomorphism onto the open subset $C^r_c(M,V_\phi) \subseteq C^r_c(M,E)$, for some chart $\phi\colon U_\phi \to V_\phi \subseteq E$ of G around 1 such that $\phi(1) = 0$. □

The Lie groups $C^r_c(M,G)$ described in (c) are also called *test function groups*.

10 Differentiability of almost local mappings

We describe a criterion ensuring differentiability properties for mappings between open subsets of spaces of vector-valued test functions (equipped with the box topology). Cf. [32], [33], [27] and their precursor [25] for analogous results in the real locally convex case, based on the locally convex direct limit topology.

10.1 Our general setting is the following: $\mathbb{F}$ is the field of real numbers or a local field, and $\mathbb{K}$ a valued field which is a topological extension field of $\mathbb{F}$.[12] For $r,s,k \in \mathbb{N}_0 \cup \{\infty\}$, we are given a paracompact, finite-dimensional $\mathbb{F}$-manifold M of class $C^r_{\mathbb{F}}$; a paracompact, finite-dimensional $\mathbb{F}$-manifold N of class $C^s_{\mathbb{F}}$; and topological $\mathbb{K}$-vector spaces E and F. We consider a mapping $f\colon P \to C^s_c(N,F)$, defined on an open subset $P \subseteq C^r_c(M,E)$.

Our investigations are stimulated by the following question:

10.2 Question. If $f|_{P\cap C^r_K(M,E)}$ is of class $C^k_{\mathbb{K}}$ for all compact subsets $K \subseteq M$, does it follow that f is $C^k_{\mathbb{K}}$?

The answer is *negative*. For example, the self-map

$$f\colon C^\infty_c(\mathbb{R},\mathbb{R}) \to C^\infty_c(\mathbb{R},\mathbb{R}), \quad \gamma \;\mapsto\; \gamma\circ\gamma \,-\, \gamma(0)$$

of the space of real-valued test functions on the line is discontinuous at $\gamma = 0$, although $f|_{C^\infty_K(\mathbb{R},\mathbb{R})}$ is smooth, for all compact subsets $K \subseteq \mathbb{R}$ (see [31]).

The goal of this section is to describe a simple *additional condition* which prevents the type of pathology just described. As we shall see, Question 10.2 has an affirmative answer if we require in addition that f be "almost local." Being almost local is a rather mild condition, which is satisfied by most of the mappings of relevance, for example by all mappings encountered in the construction of the Lie group structure on groups of compactly supported diffeomorphisms of finite-dimensional smooth manifolds over the reals (see [33]).

[12] The case $\mathbb{F} = \mathbb{C}$ has to be excluded now, since we have to use compactly supported cut-off functions.

Definition 10.3 (a) A map $f\colon P \to C^s_c(N,F)$ (as in **10.1**) is called *almost local* if there exist locally finite covers $(U_i)_{i\in I}$ of M and $(V_i)_{i\in I}$ of N by relatively compact, open sets such that, for all $i \in I$ and $\gamma, \eta \in P$ with $\gamma|_{U_i} = \eta|_{U_i}$, we have $f(\gamma)|_{V_i} = f(\eta)|_{V_i}$.[13]

(b) A map $f : P \to C^s_c(N,F)$ is called *locally almost local* if every $\gamma \in P$ has an open neighbourhood $Q \subseteq P$ such that $f|_Q$ is almost local.

(c) In the special case where $M = N$, we call $f\colon P \to C^s_c(M,F)$ a *local* mapping if, for all $x \in M$ and $\gamma \in P$, the element $f(\gamma)(x)$ only depends on the germ of γ at x.[14] It is easy to see that every local mapping is almost local.

Cf. already [46, Defn. 14.13] for the related notion of a "local operator."

Theorem 10.4 (Smoothness Theorem) *Let $f\colon C^r_c(M,E) \supseteq P \to C^s_c(N,F)$ be a map as described in* **10.1**. *If $f_K := f|_{P\cap C^r_K(M,E)}$ is of class $C^k_{\mathbb{K}}$ for every compact subset $K \subseteq M$ and f is locally almost local, then f is of class $C^k_{\mathbb{K}}$.*

Proof. We proceed in steps.

10.5 Given $\gamma \in P$, there exists an open neighbourhood Q of γ in P such that $f|_Q$ is almost local. As γ was arbitrary, the assertion will follow if we can show that $f|_W$ is of class $C^k_{\mathbb{K}}$ for some open neighbourhood W of γ in Q. To this end, it suffices to show that the mapping $g\colon Q - \gamma \to C^s_c(N,F)$, $g(\eta) := f(\gamma + \eta) - f(\gamma)$ is of class $C^k_{\mathbb{K}}$ on some open zero-neighbourhood. As $f|_Q$ is almost local, we find locally finite covers $(U_i)_{i\in I}$ of M and $(V_i)_{i\in I}$ of N, with each U_i and V_i relatively compact and open, such that $f(\eta)|_{V_i}$ only depends on $\eta|_{U_i}$, for all $\eta \in Q$. Then apparently also $g(\eta)|_{V_i} = g(\xi)|_{V_i}$ for all $\eta, \xi \in Q - \gamma$ such that $\eta|_{U_i} = \xi|_{U_i}$, showing that also g is almost local. Furthermore, given a compact subset $K \subseteq M$, the map $g|_{(Q-\gamma)\cap C^r_K(M,E)}$ is of class $C^k_{\mathbb{K}}$, since so is the restriction of f to $Q \cap C^r_{K\cup \operatorname{supp}(\gamma)}(M,E)$. We abbreviate $R := Q - \gamma$.

10.6 We pick a locally finite open cover $(\widetilde{U}_i)_{i\in I}$ of M such that $\overline{U_i} \subseteq \widetilde{U}_i$ holds for the compact closures, for all $i \in I$; such a "thickening" exists by Lemma 8.5. For each $i \in I$, we pick a mapping $h_i \in C^r(\widetilde{U}_i, \mathbb{F})$, with compact support $K_i := \operatorname{supp}(h_i)$, which is constantly 1 on U_i (see Lemma 8.8 if $\mathbb{F}$ is a local field; the real case is standard).

10.7 By Example 8.19, the family $(\rho_i)_{i\in I}$ of restriction maps $\rho_i\colon C^r_c(M,E) \to C^r(\widetilde{U}_i, E)$ is a patchwork for $C^r_c(M,E)$. We let $\rho\colon C^r_c(M,E) \to \bigoplus_{i\in I} C^r(\widetilde{U}_i, E) =: S$ be the corresponding embedding taking η to $\sum_{i\in I} \rho_i(\eta)$. Similarly, the family $(\sigma_i)_{i\in I}$ of restriction maps $\sigma_i : C^s_c(N,F) \to C^s(V_i, F)$ is a patchwork for $C^s_c(N,F)$.

10.8 The mapping ρ being a topological embedding, we find an open 0-neighbourhood $H \subseteq S$ such that $\rho^{-1}(H) \subseteq R$. The direct sum being equipped with the box topology, after shrinking H we may assume that $H = \bigoplus_{i\in I} A_i$ for a family $(A_i)_{i\in I}$ of open 0-neighbourhoods $A_i \subseteq C^r(\widetilde{U}_i, E)$. The multiplication operator $\mu_{h_i}\colon C^r(\widetilde{U}_i, E) \to C^r_{K_i}(\widetilde{U}_i, E)$,

[13] In other words, $f(\gamma)|_{V_i}$ only depends on $\gamma|_{U_i}$.

[14] More precisely, we require $f(\gamma)(x) = f(\eta)(x)$ for all $x \in M$ and $\gamma, \eta \in P$ with the same germ at x.

$\eta \mapsto h_i \cdot \eta$ is continuous linear (in view of Lemma 1.15 and Lemma 4.12, applied with a cover of coordinate neighbourhoods, this assertion can be reduced to Lemma 4.5). Hence, we find an open zero-neighbourhood $W_i \subseteq A_i$ such that $h_i \cdot W_i \subseteq R$, where we identify $C^r_{K_i}(\widetilde{U}_i, E)$ with $C^r_{K_i}(M, E) \subseteq C^r_c(M, E)$ as a topological $\mathbb{K}$-vector space in the natural way, extending functions by 0 (cf. Lemma 4.24). Then $W := \rho^{-1}(\bigoplus_{i \in I} W_i) \subseteq R$ is an open zero-neighbourhood in $C^r_c(M, E)$ such that $\rho_i(W) \subseteq W_i$ for each $i \in I$. We define

$$g_i \colon W_i \to C^s(V_i, F), \quad g_i := \sigma_i \circ g|_{R \cap C^r_{K_i}(M,E)} \circ \mu_{h_i}|^R_{W_i}.$$

Then g_i is of class $C^k_{\mathbb{K}}$, being a composition of $C^k_{\mathbb{K}}$-maps. Note that $\sigma_i(g(\eta)) = g(\eta)|_{V_i} = g(h_i \cdot \eta)|_{V_i} = g_i(\eta|_{\widetilde{U}_i})$ for each $\eta \in W$ and $i \in I$. Thus $(g_i)_{i \in I}$ is compatible with $g|_W$ in the sense of Definition 8.20. We have shown that $g|_W$ is a patched mapping which is of class $C^k_{\mathbb{K}}$ on the patches. By Proposition 8.21, $g|_W$ is of class $C^k_{\mathbb{K}}$. □

11 Smoothness of evaluation and composition

We discuss differentiability properties of evaluation and composition of maps.

Proposition 11.1 *Let $\mathbb{K}$ be a locally compact topological field, $k \in \mathbb{N}_0 \cup \{\infty\}$, M a finite-dimensional $C^k_{\mathbb{K}}$-manifold, and E a topological $\mathbb{K}$-vector space. Then the "evaluation map"*

$$\varepsilon \colon C^k(M, E) \times M \to E, \quad \varepsilon(\gamma, x) := \gamma(x)$$

is of class $C^k_{\mathbb{K}}$.

Proof. Given $x \in M$, let $\kappa \colon U \to V$ be a chart of M around x, where V is an open subset of the modeling space Z of M. Then $\varepsilon(\gamma, \kappa^{-1}(y)) = (\gamma \circ \kappa^{-1})(y) = \tilde{\varepsilon}(C^k(\kappa^{-1}, E)(\gamma), y)$ for all $y \in V$, in terms of the evaluation map $\tilde{\varepsilon} \colon C^k(V, E) \times V \to E$ and the pullback $C^k(\kappa^{-1}, E)\colon C^k(M, E) \to C^k(V, E)$ which is continuous linear and thus smooth (Lemma 4.11). It therefore suffices to consider the case where $M = V$ is an open subset of a finite-dimensional $\mathbb{K}$-vector space Z. The inclusion map $C^\infty(V, E) \to C^k(V, E)$ being continuous linear for all $k \in \mathbb{N}_0$ (Remark 4.2 (a)), it also suffices to consider finite k. We proceed by induction.

The case $k = 0$ is well known (see, *e.g.*, [16], Thm. 3.4.3 and Prop. 2.6.11).

Induction step. Given $k \in \mathbb{N}$, suppose that the assertion of the lemma holds if k is replaced with $k - 1$. Given $(\gamma, x, \eta, y, t) \in (C^k(V, E) \times V)^{[1]}$ such that $t \neq 0$, we calculate

$$\begin{aligned} \tfrac{1}{t}(\varepsilon(\gamma + t\eta, x + ty) - \varepsilon(\gamma, x)) &= \tfrac{1}{t}(\gamma(x + ty) - \gamma(x)) + \eta(x + ty) \\ &= \gamma^{[1]}(x, y, t) + \eta(x + ty) . \end{aligned} \tag{21}$$

Let $\varepsilon_1 : C^{k-1}(V^{[1]}, E) \times V^{[1]} \to E$ denote the evaluation map, which is of class $C^{k-1}_{\mathbb{K}}$ by induction. Then, using Remark 4.2 (b),

$$\theta \colon (C^k(V, E) \times V)^{[1]} \to E, \quad \theta(\gamma, x, \eta, y, t) := \varepsilon(\eta, x + ty) + \varepsilon_1(\gamma^{[1]}, (x, y, t))$$

is a mapping of class $C^{k-1}_{\mathbb{K}}$. In view of (21), we deduce that ε is of class $C^1_{\mathbb{K}}$, with $\varepsilon^{[1]} = \theta$ of class $C^{k-1}_{\mathbb{K}}$, and thus ε is of class $C^k_{\mathbb{K}}$, which completes the inductive proof. $\square$

Let us turn to the composition map now. We shall show:

Proposition 11.2 *Let $\mathbb{K}$ be a locally compact topological field, E a topological $\mathbb{K}$-vector space, $r, k \in \mathbb{N}_0 \cup \{\infty\}$, M be a finite-dimensional $C^r_{\mathbb{K}}$-manifold, F a finite-dimensional $\mathbb{K}$-vector space, $U \subseteq F$ be open, and $K \subseteq M$ compact. Then the composition map*

$$\Gamma\colon C^{r+k}(U,E) \times C^r_K(M,U) \to C^r(M,E), \quad \Gamma(\gamma,\eta) := \gamma \circ \eta$$

is of class $C^k_{\mathbb{K}}$. If $k \geq 1$, then

$$d\Gamma(\gamma,\eta,\gamma_1,\eta_1) = d\gamma \circ (\eta,\eta_1) + \gamma_1 \circ \eta \tag{22}$$

for all $\gamma, \gamma_1 \in C^{r+k}(U,E)$, $\eta \in C^r_K(M,U)$, and $\eta_1 \in C^r_K(M,F)$.

If $U = F$, then we need not assume that M be finite-dimensional. In this case, we have:

Proposition 11.3 *Let $\mathbb{K}$ be a locally compact topological field, E a topological $\mathbb{K}$-vector space, $r, k \in \mathbb{N}_0 \cup \{\infty\}$, M a $C^r_{\mathbb{K}}$-manifold, modeled on an arbitrary topological $\mathbb{K}$-vector space, and F be a finite-dimensional $\mathbb{K}$-vector space. Then the composition map*

$$\Gamma\colon C^{r+k}(F,E) \times C^r(M,F) \to C^r(M,E), \quad \Gamma(\gamma,\eta) := \gamma \circ \eta$$

is of class $C^k_{\mathbb{K}}$. If $k \geq 1$, then

$$d\Gamma(\gamma,\eta,\gamma_1,\eta_1) = d\gamma \circ (\eta,\eta_1) + \gamma_1 \circ \eta \tag{23}$$

for all $\gamma, \gamma_1 \in C^{r+k}(F,E)$ and $\eta, \eta_1 \in C^r(M,F)$.

For finite-dimensional M, both propositions are immediate consequences of the following technical result, which we prove now. A direct proof for Proposition 11.3 (including the case of infinite-dimensional M) is given in Appendix C.

Lemma 11.4 *Let $\mathbb{K}$ be a locally compact topological field, E a topological $\mathbb{K}$-vector space, $r, k \in \mathbb{N}_0 \cup \{\infty\}$, M a finite-dimensional $C^r_{\mathbb{K}}$-manifold, F a finite-dimensional $\mathbb{K}$-vector space, U an open subset of F, K a compact subset of M, and $Y \subseteq K^0$ be a non-empty, open subset. Let H be a finite-dimensional $\mathbb{K}$-vector space, and $P \subseteq H$ be open. Then*

$$\Theta\colon C^{r+k}(U \times P, E) \times \lfloor K, U \rfloor_r \times P \to C^r(Y,E), \quad \Theta(\gamma,\eta,p) := \gamma({\scriptstyle\bullet},p) \circ \eta|_Y,$$

where $\lfloor K, U \rfloor_r \subseteq C^r(M,F)$, is a mapping of class $C^k_{\mathbb{K}}$. If $k \geq 1$, then

$$\begin{aligned}&\Theta^{[1]}((\gamma,\eta,p),\,(\gamma_1,\eta_1,p_1),\,t)\\ &\quad = \gamma^{[1]}(({\scriptstyle\bullet},p),\,({\scriptstyle\bullet},p_1),\,t) \circ (\eta,\eta_1)|_Y + \gamma_1({\scriptstyle\bullet},p+tp_1) \circ (\eta+t\eta_1)|_Y\end{aligned} \tag{24}$$

for all $((\gamma,\eta,p),\,(\gamma_1,\eta_1,p_1),\,t)\in(C^{r+k}(U\times P,E)\times\lfloor K,U\rfloor_r\times P)^{[1]}$.

Hence, as a special case, the map

$$\Gamma\colon C^{r+k}(U,E)\times\lfloor K,U\rfloor_r\to C^r(Y,E),\quad \Gamma(\gamma,\eta):=\gamma\circ\eta|_Y$$

is of class $C^k_{\mathbb{K}}$*. If* $k\geq 1$*, then*

$$\Gamma^{[1]}((\gamma,\eta),\,(\gamma_1,\eta_1),\,t)=\gamma^{[1]}(\bullet,t)\circ(\eta,\eta_1)|_Y+\gamma_1\circ(\eta+t\eta_1)|_Y \tag{25}$$

for all $((\gamma,\eta),\,(\gamma_1,\eta_1),\,t)\in(C^{r+k}(U,E)\times\lfloor K,U\rfloor_r)^{[1]}$*. In particular,*

$$d\Gamma((\gamma,\eta),\,(\gamma_1,\eta_1))=d\gamma\circ(\eta,\eta_1)|_Y+\gamma_1\circ\eta|_Y \tag{26}$$

for all $\gamma,\gamma_1\in C^{r+k}(U,E)$*,* $\eta\in\lfloor K,U\rfloor_r$*, and* $\eta_1\in C^r(M,F)$*.*

Remark 11.5 In the real or complex locally convex case, the desired properties of Γ can be established directly, without recourse to parameters. In the general case envisaged here, a direct induction without parameter sets (based on Proposition 11.1 and Lemma 12.1) would only show that Γ is $C^k_{\mathbb{K}}$ when $C^{r+k}(U,E)$ is replaced with $C^{r+\frac{1}{2}k(k+1)}(U,E)$, due to the loss in the order of differentiability in Lemma 12.1.

Proof of Lemma 11.4. Clearly, we only need to prove the assertions concerning Θ: then also Γ will have the asserted properties. It suffices to consider finite $k\in\mathbb{N}_0$ (cf. Remark 4.2 (a)). We may also assume that $r\in\mathbb{N}_0$ (cf. proof of Proposition 4.16). Thus, we assume that both r and k are finite, and prove the assertion by induction on k.

The case $k=0$

We proceed by induction on r. Let us suppose that $r=0$ first. We recall that the topology we have defined on spaces of C^0-maps coincides with the compact-open topology (Remark 4.10). For $(\gamma,\eta,p)\in C(U\times P,E)\times\lfloor K,U\rfloor\times P$, we have

$$\Theta(\gamma,\eta,p)=\widetilde{\Gamma}\,(\gamma,\ \eta|_Y\times\mathrm{id}_P)\,(\bullet,p)\,,\quad \text{where} \tag{27}$$

$$\widetilde{\Gamma}\colon C(U\times P,\,E)_{c.o.}\times C(Y\times P,\,U\times P)_{c.o.}\to C(Y\times P,\,E)_{c.o.},\quad \widetilde{\Gamma}(\sigma,\tau):=\sigma\circ\tau$$

is the composition map, which is continuous since $U\times P$ is locally compact [16, Thm. 3.4.2]. It easily follows from the definition of the compact-open topology that the mapping $\lfloor K,U\rfloor\to C(Y\times P,U\times P)_{c.o.}$, $\eta\mapsto\eta|_Y\times\mathrm{id}_P$ is continuous. The map

$$f^\vee:\ P\to C(Y,E)\,,\qquad f^\vee(p):=f(\bullet,p)$$

is continuous for $f\in C(Y\times P,E)$, and also the map $C(Y\times P,E)\to C(P,C(Y,E))$, $f\mapsto(f^\vee:\ p\mapsto f(\bullet,p))$ is continuous [16, Thm. 3.4.7]. Furthermore, P being locally compact, the evaluation map $\varepsilon:\ C(P,C(Y,E))\times P\to C(Y,E)$ is continuous (cf. [16], Thm. 3.4.3 and Prop. 1.6.11). Reading (27) as $\Theta(\gamma,\eta,p)=\varepsilon(\widetilde{\Gamma}(\gamma,\ \eta|_Y\times\mathrm{id}_P)^\vee,\ p)$, we see

that Θ is continuous.

Induction step on r. Let $r \in \mathbb{N}$, and suppose that the proposition holds for $k=0$, when r is replaced with $r-1$. It then suffices to show continuity of Θ in the case where M is an open subset of its modeling space Z. In fact, suppose that M is a $C^r_{\mathbb{K}}$-manifold. For each $y \in Y$, there exists a chart $\kappa_y \colon W_y \to V_y \subseteq Z$ of Y around y. Let $L_y \subseteq W_y$ be a compact neighbourhood of y, and $K_y := \kappa_y(L_y)$; let $Y_y := K_y^0$ be the interior of K_y. Since $(L_y^0)_{y\in Y}$ is an open cover of Y, we deduce with Lemma 4.12 that Θ will be continuous if we can show that

$$h_y \colon C^r(U\times P,E)\times \lfloor K,U\rfloor_r \times P \to C^r(Y_y,E), \quad h_y(\gamma,\eta,p) := \Theta(\gamma,\eta,p)\circ \kappa_y^{-1}|_{Y_y}$$

is continuous, for all $y \in Y$. But

$$h_y(\gamma,\eta,p) = \gamma(\bullet,p)\circ(\eta\circ\kappa_y^{-1})|_{Y_y} = \Theta_y(\gamma,\eta\circ\kappa_y^{-1},p) \tag{28}$$

with $\Theta_y \colon C^r(U\times P,E)\times\lfloor K_y,U\rfloor_r\times P \to C^r(Y_y,E)$, $\Theta_y(\gamma,\sigma,p) := \gamma(\bullet,p)\circ\sigma|_{Y_y}$, where $\lfloor K_y,U\rfloor_r \subseteq C^r(V_y,F)$. Note that the pullback $C^r(M,F)\to C^r(V_y,F)$, $\eta\mapsto \eta\circ\kappa_y^{-1}$ is continuous linear (Lemma 4.11) and takes the open set $\lfloor K,U\rfloor_r$ into $\lfloor K_y,U\rfloor_r$. Thus (28) shows that h_y will be continuous if each Θ_y is continuous. Since V_y is open in Z, this completes the reduction step to the case where M is open in Z.

To complete the induction step on r in the case $k=0$, by the preceding we may assume now that M is an open subset of Z. The map $\Theta\colon C^r(U\times P,E)\times\lfloor K,U\rfloor_r\times P\to C^r(Y,E)$ is continuous as a map into $C(Y,E)$, by the case $r=0$ already settled and Remark 4.2 (a). Hence, in view of Remark 4.2 (b), Θ will be continuous if we can show that the map

$$C^r(U\times P,E)\times\lfloor K,U\rfloor_r\times P\to C^{r-1}(Y^{[1]},E), \quad (\gamma,\eta,p)\mapsto\Theta(\gamma,\eta,p)^{[1]}$$

is continuous at each given element (γ_0,η_0,p_0) in its domain, where

$$\Theta(\gamma,\eta,p)^{[1]}(x,y,t) = \gamma^{[1]}((\eta(x),p),\,(\eta^{[1]}(x,y,t),0),\,t) \tag{29}$$

for all $(x,y,t)\in Y^{[1]}$, by the Chain Rule. Let $(x_0,y_0,t_0)\in Y^{[1]}$ be given. There exist open neighbourhoods $U_1\subseteq U$ of $\eta_0(x_0)$, $U_2\subseteq F$ of $\eta_0^{[1]}(x_0,y_0,t_0)$ and $U_3\subseteq\mathbb{K}$ of t_0, such that $U_1\times P\times U_2\times\{0\}\times U_3\subseteq(U\times P)^{[1]}$. Then

$$\rho\colon C^{r-1}((U\times P)^{[1]},E)\to C^{r-1}(U_1\times U_2\times U_3\times P,E), \quad \rho(\xi)(x,y,t,p) := \xi(x,p,y,0,t)$$

is a continuous linear map (Lemma 4.11). There exist open neighbourhoods $V_1\subseteq Y$ of x_0, $V_2\subseteq Z$ of y_0, and $V_3\subseteq U_3$ of t_0 such that $\eta_0(V_1)\subseteq U_1$, $V_1\times V_2\times V_3\subseteq Y^{[1]}$, and $\eta_0^{[1]}(V_1\times V_2\times V_3)\subseteq U_2$. There exist compact neighbourhoods $K_1\subseteq V_1$ of x_0, $K_2\subseteq V_2$ of y_0, and $K_3\subseteq V_3$ of t_0. Set $Y_i := K_i^0$ for $i=1,2,3$. By induction, the map

$$\widetilde{\Theta}\colon C^{r-1}(U_1\times U_2\times U_3\times P,E)\times\lfloor K_1\times K_2\times K_3,U_1\times U_2\times U_3\rfloor_{r-1}\times P\to C^{r-1}(Y_1\times Y_2\times Y_3,E)$$

taking (σ,τ,p) to $\sigma(\bullet,p)\circ\tau|_{Y_1\times Y_2\times Y_3}$ is continuous; here $\lfloor K_1\times K_2\times K_3, U_1\times U_2\times U_3\rfloor_{r-1}\subseteq C^{r-1}(V_1\times V_2\times V_3, F\times F\times\mathbb{K})$. Note that

$$\Omega := \big\{\eta\in\lfloor K,U\rfloor_r\cap\lfloor K_1,U_1\rfloor_r\colon \eta^{[1]}|_{V_1\times V_2\times V_3}\in\lfloor K_1\times K_2\times K_3, U_2\rfloor_{r-1}\big\}$$

is an open neighbourhood of η_0 in $\lfloor K,U\rfloor_r$. Furthermore, by Lemma 4.11, the map

$$h\colon\Omega\to\lfloor K_1\times K_2\times K_3, U_1\times U_2\times U_3\rfloor_{r-1},\quad h(\eta):=\big(\eta\circ\pi_1,\ \eta^{[1]}|_{V_1\times V_2\times V_3},\ \pi_3\big)$$

is continuous, where $\pi_1 : V_1\times V_2\times V_3\to V_1$ and $\pi_3 : V_1\times V_2\times V_3\to V_3\subseteq U_3$ are the coordinate projections. Since, by (29) and the definition of ρ and h, we have

$$\Theta(\gamma,\eta,p)^{[1]}|_{Y_1\times Y_2\times Y_3}=\widetilde{\Theta}(\rho(\gamma^{[1]}),h(\eta))$$

for all $p\in P$, $\gamma\in C^r(U\times P,E)$, and $\eta\in\Omega$, we see that $(\gamma,\eta,p)\mapsto\Theta(\gamma,\eta,p)^{[1]}|_{Y_1\times Y_2\times Y_3}\in C^{r-1}(Y_1\times Y_2\times Y_3,E)$ is continuous at (γ_0,η_0,p_0). Since $Y^{[1]}$ can be covered by sets of the form $Y_1\times Y_2\times Y_3$ as before, using Lemma 4.12 we now deduce that the mapping $(\gamma,\eta,p)\mapsto\Theta(\gamma,\eta,p)^{[1]}\in C^{r-1}(Y^{[1]},E)$ is continuous at (γ_0,η_0,p_0), as desired.

Induction step on k

Let $k\in\mathbb{N}$, and suppose that the assertion of the lemma holds when k is replaced with $k-1$, for all $r\in\mathbb{N}_0$. Let $r\in\mathbb{N}_0$. Given an element $((\gamma,\eta,p),(\gamma_1,\eta_1,p_1),t)\in\Omega:=(C^{r+k}(U\times P,E)\times\lfloor K,U\rfloor_r\times P)^{[1]}$ such that $t\neq 0$, we calculate for $x\in Y$:

$$\begin{aligned}
&\tfrac{1}{t}\left(\Theta(\gamma+t\gamma_1,\eta+t\eta_1,p+tp_1)-\Theta(\gamma,\eta,p)\right)(x)\\
&\quad=\ \tfrac{1}{t}\Big(\gamma(\eta(x)+t\eta_1(x),p+tp_1)-\gamma(\eta(x),p)\Big)\ +\ \gamma_1(\eta(x)+t\eta_1(x),p+tp_1)\\
&\quad=\ \gamma^{[1]}((\eta(x),p),\,(\eta_1(x),p_1),\,t)\ +\ \gamma_1(\eta(x)+t\eta_1(x),p+tp_1)\,,
\end{aligned}\tag{30}$$

in accordance with (24). Since Θ is $C^{k-1}_{\mathbb{K}}$ and hence continuous as a consequence of the induction hypothesis, in order that Θ be $C^k_{\mathbb{K}}$, it therefore only remains to show that the mapping $\Omega\to C^r(Y,E)$ described in (24), let us call it g, is of class $C^{k-1}_{\mathbb{K}}$ (then $g=\Theta^{[1]}$). Since Θ is $C^{k-1}_{\mathbb{K}}$, the map g is $C^{k-1}_{\mathbb{K}}$ on an open neighbourhood of each given element $((\bar\gamma,\bar\eta,\bar p),(\bar\gamma_1,\bar\eta_1,\bar p_1),\bar t)\in\Omega$, provided $\bar t\neq 0$. It remains to consider the case where $\bar t=0$. There is a balanced, open zero-neighbourhood $W\subseteq F$ such that $\bar\eta(K)+W+W+W\subseteq U$. Next, there are open neighbourhoods $P_0\subseteq P$ of $\bar p$, $P_1\subseteq H$ of $\bar p_1$, and $r\in\,]0,1]$ such that

$$P_0+P_2P_1\subseteq P\quad\text{and thus}\quad P_0\times P_1\times P_2\subseteq P^{[1]}\,,$$

where $P_2:=\{t\in\mathbb{K}:|t|\leq r\}$. After shrinking r, we may assume that furthermore $P_2\cdot\bar\eta_1(K)\subseteq W$. We let $U_0:=\bar\eta(K)+W\subseteq U$ and $U_1:=\bar\eta_1(K)+W$. Then

$$U_0+P_2U_1\subseteq\bar\eta(K)+W+P_2\bar\eta_1(K)+P_2W\subseteq\bar\eta(K)+W+W+W\subseteq U$$

and hence $U_0\times U_1\times P_2\subseteq U^{[1]}$. Furthermore, we have

$$(\bar\eta,\bar\eta_1)\in\lfloor K,U_0\times U_1\rfloor_r\subseteq C^r(M,F\times F)\,.$$

Then $U_0 \times P_0 \times U_1 \times P_1 \times P_2 \subseteq (U \times P)^{[1]}$, and the map

$$\begin{aligned} \rho\colon C^{r+k}(U,E) &\to C^{r+(k-1)}((U_0 \times U_1) \times (P_0 \times P_1 \times P_2), E),\\ \rho(\gamma)((u_0,u_1),\,(p,p_1,t)) &:= \gamma^{[1]}((u_0,p),\,(u_1,p_1),\,t) \end{aligned}$$

is continuous linear by Remark 4.2 (b) and Lemma 4.11. Hence ρ is $C^{k-1}_{\mathbb{K}}$. By the induction hypothesis, the map

$$\widetilde{\Theta}\colon C^{r+(k-1)}((U_0 \times U_1) \times (P_0 \times P_1 \times P_2), E) \times \lfloor K, U_0 \times U_1 \rfloor_r \times (P_0 \times P_1 \times P_2) \to C^r(Y,E)$$

taking $(\xi, \zeta, (p,p_1,t))$ to $\xi(\bullet, (p,p_1,t)) \circ \zeta$ is of class $C^{k-1}_{\mathbb{K}}$. The set

$$C^{r+k}(U \times P, E) \times \lfloor K, U_0 \rfloor_r \times P_0 \times C^{r+k}(U \times P, E) \times \lfloor K, U_1 \rfloor_r \times P_1 \times P_2$$

is an open neighbourhood of $(\bar{\gamma}, \bar{\eta}, \bar{p}, \bar{\gamma}_1, \bar{\eta}_1, \bar{p}_1, 0)$ in the domain Ω of g. For all elements $(\gamma, \eta, p, \gamma_1, \eta_1, p_1, t)$ in this open neighbourhood, we have

$$g(\gamma,\eta,p,\gamma_1,\eta_1,p_1,t) = \widetilde{\Theta}(\rho(\gamma),\,(\eta,\eta_1),\,(p,p_1,t)) \;+\; \Theta(\gamma_1, \eta + t\eta_1, p + tp_1)\,,$$

showing that g is $C^{k-1}_{\mathbb{K}}$ on this open neighbourhood. This completes the proof. □

Proposition 11.2 and Proposition 11.3 (for finite-dimensional M) now readily follow:

Proof of Proposition 11.3 for finite-dimensional M. Every $x \in M$ has a compact neighbourhood K_x; let $Y_x := K_x^0$ be its interior. Then $(Y_x)_{x \in M}$ is an open cover of M. Let $\rho_x : C^r(M,E) \to C^r(Y_x,E)$, $\gamma \mapsto \gamma|_{Y_x}$ be the restriction map. Then, as a consequence of Lemma 1.15 and Lemma 4.12, the composition map $\Gamma\colon C^{r+k}(F,E) \times C^r(M,F) \to C^r(M,E)$ will be of class C^k if we can show that

$$\rho_x \circ \Gamma\colon C^{r+k}(F,E) \times C^r(M,F) \to C^r(Y_x,E)$$

is of class C^s, for each $x \in M$. However, we have $\rho_x \circ \Gamma = \Gamma_x$, where

$$\Gamma_x\colon C^{r+k}(F,E) \times \lfloor K_x, F \rfloor_r \to C^r(Y_x,E), \quad \Gamma_x(\gamma,\eta) := \gamma \circ (\eta|_{Y_x})$$

is of class C^k by Lemma 11.4. Hence Γ is of class C^k. Suppose that $k \geq 1$ now. The mapping ρ_x being continuous linear, we have $d\Gamma_x = d(\rho_x \circ \Gamma) = \rho_x \circ d\Gamma$. Hence (26) implies that, for all $\gamma, \gamma_1 \in C^{r+k}(F,E)$, $\eta, \eta_1 \in C^r(M,F)$:

$$d\Gamma((\gamma,\eta),\,(\gamma_1,\eta_1))|_{Y_x} = d\gamma \circ (\eta,\eta_1)|_{Y_x} + \gamma_1 \circ \eta|_{Y_x} = (d\gamma \circ (\eta,\eta_1) + \gamma_1 \circ \eta)\,|_{Y_x}$$

for all $x \in M$, entailing that $d\Gamma((\gamma,\eta),\,(\gamma_1,\eta_1)) = d\gamma \circ (\eta,\eta_1) + \gamma_1 \circ \eta$, as asserted. □

Proof of Proposition 11.2. Let K_x, Y_x and $\rho_x : C^r(M,E) \to C^r(Y_x,E)$ be as in the preceding proof. In order that the composition map

$$\Gamma\colon C^{r+k}(U,E) \times C^r_K(M,U) \to C^r(M,E)$$

be of class C^k, we only need to show that $\rho_x \circ \Gamma$ is of class C^k for all $x \in M$. But $\rho_x \circ \Gamma = \Gamma_x \circ \lambda_x$, where

$$\Gamma_x \colon C^{r+k}(U,E) \times \lfloor K_x, U \rfloor_r \to C^r(Y_x, E), \quad \Gamma_x(\gamma,\eta) := \gamma \circ (\eta|_{Y_x}^U)$$

is of class C^k by Lemma 11.4, and

$$\lambda_x \colon C^{r+k}(U,E) \times C^r_K(M,U) \to C^{r+k}(U,E) \times \lfloor K_x, U \rfloor_r, \quad (\gamma,\eta) \mapsto (\gamma,\eta)$$

is obtained by restricting and co-restricting a continuous linear map to open sets and therefore smooth. Hence $\rho_x \circ \Gamma = \Gamma_x \circ \lambda_x$ is of class C^k, being a composition of C^k-maps. The desired formula for $d\Gamma$ (if $k \geq 1$) can now be deduced as in the preceding proof. □

12 Basic exponential law for smooth mappings

In this section, we establish an exponential law for smooth mappings on products of suitable manifolds, and related results.

Lemma 12.1 *Let $\mathbb{K}$ be a topological field, $r, k \in \mathbb{N}_0 \cup \{\infty\}$, M and N be $C^{r+k}_{\mathbb{K}}$-manifolds modeled on topological $\mathbb{K}$-vector spaces, and E be a topological $\mathbb{K}$-vector space. Then the following holds:*

(a) *For each mapping $f \colon M \times N \to E$ of class $C^{r+k}_{\mathbb{K}}$, the associated mapping*

$$f^\vee \colon M \to C^r(N,E), \quad f^\vee(x) := f(x, \bullet)$$

is of class $C^k_{\mathbb{K}}$.

(b) *The linear map $\Phi \colon C^{r+k}(M \times N, E) \to C^k(M, C^r(N,E))$, $\Phi(f) := f^\vee$ is continuous.*

Proof. The lemma will hold in general if we can prove the case where M and N are open subsets of topological $\mathbb{K}$-vector spaces X and Y, respectively. In fact, suppose that M and N are $C^{r+k}_{\mathbb{K}}$-manifolds. Let $f \colon M \times N \to E$ be a $C^{r+k}_{\mathbb{K}}$-map. The mapping $f^\vee$ will be of class $C^k_{\mathbb{K}}$ if we can show that it is $C^k_{\mathbb{K}}$ on some open neighbourhood of each given point $x_0 \in M$. Given x_0, we let $\phi \colon U_\phi \to V_\phi \subseteq X$ be a chart of M around x_0. Let $\mathcal{A}$ be an atlas for N, of charts $\psi \colon U_\psi \to V_\psi \subseteq Y$ of N. As a consequence of Lemma 1.15, Lemma 4.12 and Lemma 4.11, the map $f^\vee|_{U_\phi}$ is $C^k_{\mathbb{K}}$ if and only if

$$C^r(\psi^{-1}, E) \circ f^\vee|_{U_\phi} \colon U_\phi \to C^r(V_\psi, E), \quad x \mapsto f^\vee(x) \circ \psi^{-1} \tag{31}$$

is $C^k_{\mathbb{K}}$ for each $\psi \in \mathcal{A}$. This holds if and only if

$$C^r(\psi^{-1}, E) \circ f^\vee \circ \phi^{-1} \colon V_\phi \to C^r(V_\psi, E)$$

is $C^k_{\mathbb{K}}$, for each $\psi \in \mathcal{A}$. Now, for given ψ, the latter map coincides with $g^\vee$, where $g := f \circ (\phi^{-1} \times \psi^{-1}) \colon V_\phi \times V_\psi \to E$, and here $V_\phi \subseteq X$ and $V_\psi \subseteq Y$ are open subsets of topological

$\mathbb{K}$-vector spaces. It therefore suffices to show that each $g^\vee$ is of class $C^k_{\mathbb{K}}$.

To see that also (b) can be reduced to the case of open subsets of topological vector spaces, note that, as a consequence of Lemma 4.11, Lemma 4.12, Lemma 4.13 and Lemma 4.14, the topology on $C^k(M, C^r(N,E))$ is initial with respect to the family of mappings

$$h_{\phi,\psi} := C^k(V_\phi, C^r(\psi^{-1}, E)) \circ C^k(\phi^{-1}, C^r(N,E)) \colon C^k(M, C^r(N,E)) \to C^k(V_\phi, C^r(V_\psi, E))$$

taking $g \in C^k(M, C^r(N,E))$ to $C^r(\psi^{-1}, E) \circ g \circ \phi^{-1}$, where ϕ and ψ range through the charts of M and N, respectively. Since $h_{\phi,\psi}(f^\vee) = (f \circ (\phi^{-1} \times \psi^{-1}))^\vee = \big(C^{r+k}(\phi^{-1} \times \psi^{-1}, E)(f)\big)^\vee$, where $C^{r+k}(\phi^{-1} \times \psi^{-1}, E) \colon C^{r+k}(M \times N, E) \to C^{r+k}(V_\phi \times V_\psi, E)$ is continuous and takes f to a mapping defined on the product $V_\phi \times V_\psi$ of open subsets of X and Y, it suffices to prove (b) for mappings on such products $V_\phi \times V_\psi$.

By the preceding, we may assume for the rest of the proof that $U := M \subseteq X$ and $V := N \subseteq Y$ are open subsets of topological vector spaces. Recall from Remark 4.2 (a) that $C^\infty(V,E) = \varprojlim_{r \in \mathbb{N}_0} C^r(V,E)$. Accordingly, $C^k(U, C^\infty(V,E)) = \varprojlim_{r \in \mathbb{N}_0} C^k(U, C^r(V,E))$ (Lemma 1.17, Lemma 4.14). It therefore suffices to prove the assertions when $r \in \mathbb{N}_0$. By a similar argument, we may assume that k is finite. The proof is by induction on $k \in \mathbb{N}_0$.

The case $k = 0$. If $r = 0$, then (a) and (b) are special cases of [16], Thm. 3.4.1 and 3.4.7, respectively. To proceed by induction on r, suppose that $r \in \mathbb{N}$, and suppose the assertion of the lemma holds when r is replaced with $r-1$. The topology on $C^r(V,E)$ is initial with respect to the maps $\alpha \colon C^r(V,E) \to C^{r-1}(V,E)$, $\gamma \mapsto \gamma$ and $\beta \colon C^r(V,E) \to C^{r-1}(V^{[1]}, E)$, $\beta(\gamma) := \gamma^{[1]}$ (Remark 4.2). Hence the topology on $C(U, C^r(V,E))$ is initial with respect to the mappings $C(U,\alpha)$ and $C(U,\beta)$ (Lemma 4.14).

(a) Let $f \colon U \times V \to E$ be a $C^r_{\mathbb{K}}$-map, and $f^\vee \colon U \to C^r(V,E)$ be as above. By the induction hypothesis, $U \to C^{r-1}(V,E)$, $x \mapsto f(x, \bullet) = \alpha \circ f^\vee$ is a continuous mapping. In view of the preceding, $f^\vee \colon U \to C^r(V,E)$ will be continuous if we can show that also

$$\beta \circ f^\vee \colon U \to C^{r-1}(V^{[1]}, E), \quad x \mapsto (f^\vee(x))^{[1]}$$

is continuous. However,

$$\psi_f \colon U \times V^{[1]} \to E, \;\; \psi_f(x,(v,h,t)) := f^{[1]}((x,v),\, (0,h),\, t) \text{ for } x \in U,\ (v,h,t) \in V^{[1]} \quad (32)$$

is of class $C^{r-1}_{\mathbb{K}}$, being a partial map of $f^{[1]}$. Since $(\beta \circ f^\vee)(x) = \psi_f(x, \bullet) = (\psi_f)^\vee(x)$, the map $\beta \circ f^\vee = (\psi_f)^\vee$ is continuous by the induction hypothesis. Thus $f^\vee$ is continuous, and thus (a) holds in the $C^r_{\mathbb{K}}$-case, when $k = 0$.

(b) As an immediate consequence of the induction hypothesis, the mapping $C(U,\alpha) \circ \Phi \colon C^r(U \times V, E) \to C(U, C^{r-1}(V,E))$, $f \mapsto (x \mapsto f(x,\bullet))$ is continuous. The mapping

$$\Psi \colon C^r(U \times V, E) \to C^{r-1}(U \times V^{[1]}, E), \quad \Psi(f) := \psi_f$$

(with ψ_f as in (32)) is continuous by Remark 4.2 (b) and Lemma 4.4. Furthermore,

$$\Xi \colon C^{r-1}(U \times V^{[1]}, E) \to C(U, C^{r-1}(V^{[1]}, E)), \quad \Xi(g)(x) := g(x, \bullet) \;\text{ for } x \in U$$

is continuous, by induction. Thus $C(U,\beta)\circ\Phi=\Xi\circ\Psi$ is continuous. The topology on $C(U,C^r(V,E))$ being initial with respect to $C(U,\alpha)$ and $C(U,\beta)$, we deduce from the preceding that $\Phi\colon C^r(U\times V,E)\to C(U,C^r(V,E))$ is continuous. Thus also the $C^r_{\mathbb{K}}$-case of (b) is established, when $k=0$.

Induction step on k. Let $k\in\mathbb{N}$, and suppose that the assertions of the lemma hold for all r, when k is replaced with $k-1$. Let $f\colon U\times V\to E$ be a mapping of class $C^{r+k}_{\mathbb{K}}$. As a consequence of the induction hypothesis and Remark 4.2 (a), the map $f^\vee\colon U\to C^r(V,E)$ is of class $C^{k-1}_{\mathbb{K}}$, and $C^{r+k}(U\times V,E)\to C^{k-1}(U,C^r(V,E))$, $f\mapsto f^\vee$ is a continuous linear map. We now observe that

$$\tfrac{1}{t}(f^\vee(x+ty)-f^\vee(x))(v)=\tfrac{1}{t}(f(x+ty,v)-f(x,v))=f^{[1]}(x,v,y,0,t) \tag{33}$$

for all $v\in V$ and $(x,y,t)\in U^{[1]}$ such that $t\neq 0$. The mapping $\delta(f)\colon U^{[1]}\times V\to E$, $\delta(f)(x,y,t,v):=f^{[1]}(x,v,y,0,t)$ is $C^{r+k-1}_{\mathbb{K}}$, and $\delta\colon C^{r+k}(U\times V,E)\to C^{r+k-1}(U^{[1]}\times V,E)$ is a continuous linear map (see Remark 4.2 (b), Lemma 4.4). By the induction hypothesis, for any $g\in C^{r+k-1}(U^{[1]}\times V,E)$, the map

$$\Psi(g):=g^\vee\colon U^{[1]}\to C^r(V,E),\quad \Psi(g)(x,y,t)(v):=g((x,y,t),\,v)\quad \text{for } (x,y,t)\in U^{[1]},\ v\in V$$

is of class $C^{k-1}_{\mathbb{K}}$, and the map $\Psi\colon C^{r+k-1}(U^{[1]}\times V,E)\to C^{k-1}(U^{[1]},C^r(V,E))$ so obtained is continuous and linear. By the preceding, given $f\in C^{r+k}(U\times V,E)$, we have $\Psi(\delta(f))\in C^{k-1}(U^{[1]},C^r(V,E))$. In particular, $\Psi(\delta(f))=\delta(f)^\vee$ is continuous. Note that (33) can be read as

$$(f^\vee)^{]1[}(x,y,t)\;=\;(\delta(f))^\vee(x,y,t)\qquad\text{for all }\ (x,y,t)\in U^{]1[}.$$

Thus $f^\vee$ is of class $C^1_{\mathbb{K}}$ with $(f^\vee)^{[1]}=\delta(f)^\vee=\Psi(\delta(f))$. Now $f^\vee$ being of class $C^1_{\mathbb{K}}$ with $(f^\vee)^{[1]}$ of class $C^{k-1}_{\mathbb{K}}$, the mapping $f^\vee$ is of class $C^k_{\mathbb{K}}$. Since Φ is continuous when considered as a mapping $C^{r+k}(U\times V,E)\to C(U,C^r(V,E))$ as a consequence of the case $k=0$, and the map $C^{r+k}(U\times V,E)\to C^{k-1}(U^{[1]},C^r(V,E))$, $f\mapsto (f^\vee)^{[1]}=(\Psi\circ\delta)(f)$ is continuous by the preceding, we deduce with Remark 4.2 (b) that $\Phi\colon C^{r+k}(U\times V,E)\to C^k(U,C^r(V,E))$, $\Phi(f)=f^\vee$ is continuous. This completes the proof. □

Proposition 12.2 *Let $\mathbb{K}$ be a locally compact topological field, E be a topological $\mathbb{K}$-vector space, M be a $C^\infty_{\mathbb{K}}$-manifold modeled on a topological $\mathbb{K}$-vector space, and N be a finite-dimensional $C^\infty_{\mathbb{K}}$-manifold. Then the following holds:*

(a) *A mapping $g\colon M\to C^\infty(N,E)$ is of class $C^\infty_{\mathbb{K}}$ if and only if*

$$g^\wedge\colon M\times N\to E,\quad g^\wedge(x,y):=g(x)(y)$$

is of class $C^\infty_{\mathbb{K}}$.

(b) *The mapping* $\quad\Phi\colon C^\infty(M\times N,E)\to C^\infty(M,C^\infty(N,E)),\quad \Phi(f):=f^\vee$

is an isomorphism of topological $\mathbb{K}$-vector spaces, with inverse given by

$$\Phi^{-1}\colon C^\infty(M,C^\infty(N,E))\to C^\infty(M\times N,E),\quad \Phi^{-1}(g)=g^\wedge\,.$$

Proof. (a) By Proposition 11.1, the evaluation map $\varepsilon\colon C^\infty(N,E)\times N\to E$ is smooth. The formula $g^\wedge=\varepsilon\circ(g\times \mathrm{id}_N)$ for $g\in C^\infty(M,C^\infty(N,E))$ shows that $g^\wedge$ is smooth whenever so is g. If, on the other hand, $g\colon M\to C^\infty(N,E)$ is a mapping such that $f:=g^\wedge$ is smooth, then $g=f^\vee$ is smooth, by Lemma 12.1 (a).

(b) As a consequence of Lemma 12.1 and Part (a) of the present proposition, the mapping Φ is an isomorphism of vector spaces and continuous, with inverse given by $\Phi^{-1}(g)=g^\wedge$ for $g\in C^\infty(M,C^\infty(N,E))$. In order that Φ^{-1} be continuous, in view of Lemma 4.12 and Lemma 4.11, we only need to show that

$$C^\infty(\mathrm{id}_M\times\psi^{-1},E)\circ\Phi^{-1}\colon C^\infty(M,C^\infty(N,E))\to C^\infty(M\times V_\psi,E),$$
$$f\mapsto \Phi^{-1}(f)\circ(\mathrm{id}_M\times\psi^{-1})=f^\wedge\circ(\mathrm{id}_M\times\psi^{-1})$$

is continuous, for each chart $\psi : U_\psi\to V_\psi\subseteq Y$ of N, where Y is the modeling space of N. Note that $f^\wedge\circ(\mathrm{id}_M\times\psi^{-1})=(C^\infty(\psi^{-1},E)\circ f)^\wedge=(C^\infty(M,C^\infty(\psi^{-1},E))(f))^\wedge$ for $f\in C^\infty(M,C^\infty(N,E))$, and thus

$$C^\infty(\mathrm{id}_M\times\psi^{-1},E)\circ\Phi^{-1}=\Psi\circ C^\infty(M,C^\infty(\psi^{-1},E))\,,$$

where $\Psi\colon C^\infty(M,C^\infty(V_\psi,E))\to C^\infty(M\times V_\psi,E)$, $g\mapsto g^\wedge$. The map $C^\infty(M,C^\infty(\psi^{-1},E))$ being continuous (Lemma 4.11, Lemma 4.13), it only remains to prove that Ψ is continuous. We fix ψ now, and write $V:=V_\psi$ for brevity.

Let $(W_i)_{i\in I}$ be an open cover of V, where $W_i\subseteq V$ is relatively compact for each $i\in I$, with compact closure $K_i:=\overline{W_i}\subseteq V$. As a consequence of Lemma 4.12, the map Ψ will be continuous if we can show that $\rho_i\circ\Psi$ is continuous for each $i\in I$, where $\rho_i\colon C^\infty(M\times V,E)\to C^\infty(M\times W_i,E)$ is the restriction map. Hold $i\in I$ fixed. We have

$$\rho_i(\Psi(g))=\rho_i(g^\wedge)=(\varepsilon\circ(g\times\mathrm{id}_V))|_{M\times W_i}\,, \tag{34}$$

where $\varepsilon\colon C^\infty(V,E)\times V\to E$ is evaluation (which is $C^\infty_{\mathbb{K}}$ by Proposition 11.1). We want to re-write (34) further in order to be able to apply Proposition 4.23 (b). To this end, we let $\sigma\colon W_i\to V$ be inclusion. We define

$$\tilde h\colon V\times V\times C^\infty(V,E)\to E,\quad \tilde h(v,y,\gamma):=\varepsilon(\gamma,y)=\gamma(y)$$

and $h:=\tilde h\circ(\sigma\times\mathrm{id}_V\times\mathrm{id}_{C^\infty(V,E)})\colon W_i\times V\times C^\infty(V,E)\to E$, $h(v,y,\gamma)=\gamma(y)$. Then (34) can be re-written as $\rho_i(\Psi(g))=C^\infty(\tau,E)\,(\phi(\mathrm{id}_V,g))$, where $\tau\colon M\times W_i\to W_i\times M$ is the coordinate flip and where

$$\phi\colon \lfloor K_i,V\rfloor_\infty\times C^\infty(M,C^\infty(V,E))\to C^\infty(W_i\times M,E),\quad \phi(f,g):=h_*(f,g)$$

is smooth by Proposition 4.23 (b); here $h_*(f,g)(y,x)=h(y,f(y),g(x))=g(x)(f(y))$ for $y\in W_i$, $x\in M$, and $\lfloor K_i,V\rfloor_\infty\subseteq C^\infty(V,Y)$. Hence $\rho_i\circ\Psi$ is smooth and thus continuous, which completes the proof of Part (b). $\square$

The remainder of this section is devoted to a variant of Proposition 12.2 for manifolds modeled on metrizable topological vector spaces. In order to prove the result efficiently,

we introduce as an auxiliary concept the notion of *conveniently* $\mathbb{K}$-*smooth* maps, inspired by the convenient differential calculus of Frölicher, Kriegl and Michor (devoted to the real or complex locally convex case).

Definition 12.3 Given a topological field $\mathbb{K}$ and topological $\mathbb{K}$-vector space E, a subset $U \subseteq E$ will be called c^∞-*open* if U is open in the final topology on E with respect to the set of all $C^\infty_{\mathbb{K}}$-maps $\gamma : I \to E$, where I is an open subset of $\mathbb{K}^n$ for some $n \in \mathbb{N}$.[15] A mapping $f : U \to F$ from a c^∞-open subset $U \subseteq E$ to a topological $\mathbb{K}$-vector space F is called *conveniently* $\mathbb{K}$-*smooth* (or also a $c^\infty_{\mathbb{K}}$-*map*) if $f \circ \gamma : I \to F$ is $C^\infty_{\mathbb{K}}$, for every $n \in \mathbb{N}$, open subset $I \subseteq \mathbb{K}^n$, and $C^\infty_{\mathbb{K}}$-map $\gamma : I \to E$ such that $\gamma(I) \subseteq U$.

Apparently every open subset $U \subseteq E$ is c^∞-open, and every $C^\infty_{\mathbb{K}}$-map is also $c^\infty_{\mathbb{K}}$. Furthermore, it is obvious that compositions of composable $c^\infty_{\mathbb{K}}$-maps are $c^\infty_{\mathbb{K}}$-maps. It does not pose any problems to develop a theory of $c^\infty_{\mathbb{K}}$-manifolds, along the lines of convenient differential calculus, but we refrain from doing so here, as we wish to focus on the $C^r_{\mathbb{K}}$-theory. For the present purposes, the following limited definition is sufficient: We call a map $f : M \to E$ from a $C^\infty_{\mathbb{K}}$-manifold to a topological $\mathbb{K}$-vector space *conveniently* $\mathbb{K}$-*smooth* (or a $c^\infty_{\mathbb{K}}$-map) if $f \circ \gamma$ is $C^\infty_{\mathbb{K}}$ for $C^\infty_{\mathbb{K}}$-maps $\gamma : \mathbb{K}^n \supseteq I \to M$, or, equivalently, if $f \circ \kappa^{-1}$ is conveniently $\mathbb{K}$-smooth for every chart κ in an $C^\infty_{\mathbb{K}}$-atlas for M.

Remark 12.4 Throughout this remark, suppose that $\mathbb{K} = \mathbb{R}$, or that $\mathbb{K}$ is an ultrametric field. Then, as a consequence of [3, Thm. 11.3 (a)] (applied to subsets of $\mathbb{K}^n$), a subset $U \subseteq E$ is c^∞-open if and only if $\gamma^{-1}(U)$ is open in $\mathbb{K}$ for every $C^\infty_{\mathbb{K}}$-curve $\gamma : \mathbb{K} \to E$. Furthermore, a map $f : U \to F$ is conveniently $\mathbb{K}$-smooth if and only if $f \circ \gamma$ is $C^\infty_{\mathbb{K}}$ for every $n \in \mathbb{N}$ and every $C^\infty_{\mathbb{K}}$-map $\gamma : \mathbb{K}^n \to E$ with image in U (defined on all of $\mathbb{K}^n$). If E is a metrizable topological $\mathbb{K}$-vector space, then a subset $U \subseteq E$ is open if and only if it is c^∞-open [3, Thm. 11.3 (a)]; in this case, a mapping $f : U \to F$ into a topological $\mathbb{K}$-vector space F is $C^\infty_{\mathbb{K}}$ if and only if it is conveniently $\mathbb{K}$-smooth (cf. [3, Thm. 12.4]).

Before we can formulate the exponential law, we need to have a second look at the evaluation map.

Lemma 12.5 *Let $\mathbb{K}$ be a topological field which is metrizable (or, more generally, a topological field such that $\mathbb{K}^n$ is a k-space for all $n \in \mathbb{N}$). Let M be a $C^\infty_{\mathbb{K}}$-manifold modeled on a topological $\mathbb{K}$-vector space, and E a topological $\mathbb{K}$-vector space. Then the evaluation map*

$$\varepsilon : C^\infty(M, E) \times M \to E, \quad \varepsilon(\gamma, x) := \gamma(x)$$

is conveniently $\mathbb{K}$-smooth.

Proof. Arguing similarly as in the proof of Proposition 11.1, we reduce to the case where M is an open subset of its modeling space Z, which we assume now. To establish the lemma, we show by induction on $k \in \mathbb{N}_0$ that $\varepsilon \circ c$ is of class $C^k_{\mathbb{K}}$, for every $C^\infty_{\mathbb{K}}$-map $c = (c_1, c_2) : I \to C^\infty(M, E) \times M$ defined on an open subset $I \subseteq \mathbb{K}^n$ for some $n \in \mathbb{N}$.

[15]Thus, we require that $\gamma^{-1}(U)$ be open in I for any γ.

The case $k=0$. Let c be as before. Since $\mathbb{K}^n$ is a k-space, so is its open subset I. Hence $\varepsilon\circ c$ will be continuous if we can show that $\varepsilon\circ c|_K\colon K\to E$ is continuous, for every compact subset $K\subseteq I$. As $c_2\colon I\to M$ is continuous, the set $L:=c_2(K)\subseteq M$ is compact. Since

$$\varepsilon(c(x))=c_1(x)(c_2(x))=\tilde{\varepsilon}(c_1(x)|_L,c_2(x))$$

for all $x\in K$, where the restriction map $C^\infty(M,E)\to C(L,E)_{c.o.}$, $\eta\mapsto\eta|_L$ is continuous (cf. Remark 4.10 & [16, p. 157, Eqn. (2)]) and the evaluation map $\tilde{\varepsilon}\colon C(L,E)_{c.o.}\times L\to E$ is continuous, we see that $\varepsilon\circ c|_K$ is continuous, as desired.

Induction step. Suppose that $k\in\mathbb{N}_0$ and suppose that $\varepsilon\circ c$ is of class $C^k_{\mathbb{K}}$, for all c as before. For all $(x,y,t)\in I^{]1[}$, we calculate

$$\begin{aligned}(\varepsilon\circ c)^{]1[}(x,y,t)&=\tfrac{1}{t}((\varepsilon\circ c)(x+ty)-(\varepsilon\circ c)(x))\\&=\tfrac{1}{t}(c_1(x+ty)-c_1(x))(c_2(x+ty))+\tfrac{1}{t}(c_1(x)(c_2(x+ty))-c_1(x)(c_2(x)))\\&=c_1^{[1]}(x,y,t)(c_2(x+ty))\;+\;c_1(x)^{[1]}(c_2(x),c_2^{[1]}(x,y,t),t)\\&=\varepsilon(c_1^{[1]}(x,y,t),c_2(x+ty))\;+\;(\tilde{\varepsilon}\circ\tilde{c})(x,y,t)\,,\end{aligned}\tag{35}$$

where $\tilde{c}\colon I^{[1]}\to C^\infty(M^{[1]},E)\times M^{[1]}$, $\tilde{c}(x,y,t):=(c_1(x)^{[1]},\,(c_2(x),c_2^{[1]}(x,y,t),t))$ is smooth (cf. Remark 4.2), and where evaluation $\tilde{\varepsilon}:C^\infty(M^{[1]},E)\times M^{[1]}\to E$ takes $C^\infty_{\mathbb{K}}$-maps on open subsets of $\mathbb{K}^m$ (for any $m\in\mathbb{N}$) to $C^k_{\mathbb{K}}$-maps, by induction. Since, trivially, also $I^{[1]}\to C^\infty(M,E)\times M$, $(x,y,t)\mapsto(c_1^{[1]}(x,y,t),c_2(x+ty))$ is $C^\infty_{\mathbb{K}}$ (cf. **1.7**), we deduce from the induction hypothesis that the map

$$g\colon I^{[1]}\to E,\quad g(x,y,t):=\varepsilon(c_1^{[1]}(x,y,t),c_2(x+ty))\;+\;(\tilde{\varepsilon}\circ\tilde{c})(x,y,t)$$

is of class $C^k_{\mathbb{K}}$ and thus continuous. Since $g|_{I^{]1[}}=(\varepsilon\circ c)^{]1[}$ by (35), we deduce that $\varepsilon\circ c$ is $C^1_{\mathbb{K}}$, with $(\varepsilon\circ c)^{[1]}=g$, and thus $\varepsilon\circ c$ is of class $C^{k+1}_{\mathbb{K}}$, which completes the induction. □

Proposition 12.6 *Let $\mathbb{K}$ be a metrizable topological field, E be a topological $\mathbb{K}$-vector space, and M, N be $C^\infty_{\mathbb{K}}$-manifolds modeled on arbitrary topological $\mathbb{K}$-vector spaces. Let $g\colon M\to C^\infty(N,E)$ be a map. Then the following holds:*

(a) *If g is a $c^\infty_{\mathbb{K}}$-map, then also $g^\wedge\colon M\times N\to E$, $g^\wedge(x,y):=g(x)(y)$ is a $c^\infty_{\mathbb{K}}$-map.*

(b) *Assume that $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}$ is an ultrametric field. If both M and N are modeled on metrizable topological $\mathbb{K}$-vector spaces, then g is $C^\infty_{\mathbb{K}}$ if and only if $g^\wedge$ is $C^\infty_{\mathbb{K}}$.*

(c) *$\Phi\colon C^\infty(M\times N,E)\to C^\infty(M,C^\infty(N,E))$, $f\mapsto f^\vee$ is a continuous isomorphism of vector spaces in the situation of* (b)*, whose inverse $g\mapsto g^\wedge$ is a $c^\infty_{\mathbb{K}}$-map.*

Proof. (a) Suppose that g is a $c^\infty_{\mathbb{K}}$-map. Then $g^\wedge=\varepsilon\circ(g\times\mathrm{id}_N)$, where the evaluation map $\varepsilon\colon C^\infty(N,E)\times N\to E$ is $c^\infty_{\mathbb{K}}$ by Lemma 12.5. If $\gamma=(\gamma_1,\gamma_2)\colon\mathbb{K}^n\supseteq I\to M\times N$ is a $C^\infty_{\mathbb{K}}$-map, then $g^\wedge\circ\gamma=\varepsilon\circ(g\circ\gamma_1,\gamma_2)$ is of class $C^\infty_{\mathbb{K}}$ since $(g\circ\gamma_1,\gamma_2)\colon I\to C^\infty(N,E)\times N$

is a $C^\infty_{\mathbb{K}}$-map and ε is $c^\infty_{\mathbb{K}}$. Thus $g^\wedge$ is $c^\infty_{\mathbb{K}}$.

(b) Since M and $M \times N$ are $C^\infty_{\mathbb{K}}$-manifolds modeled on metrizable topological vector spaces, where $\mathbb{K}$ is $\mathbb{R}$ or an ultrametric field, mappings on these manifolds are $C^\infty_{\mathbb{K}}$ if and only if they are $c^\infty_{\mathbb{K}}$ (cf. Remark 12.4). Hence (b) readily follows from (a) and Proposition 12.1 (a).

(c) It is immediate from (b) and Proposition 12.1 (b) that Φ is a continuous linear bijection, with inverse $\Psi : C^\infty(M, C^\infty(N,E)) \to C^\infty(M \times N, E)$, $\Psi(g) = g^\wedge$. To see that Ψ is a $c^\infty_{\mathbb{K}}$-map, let $\gamma : \mathbb{K}^n \supseteq I \to C^\infty(M, C^\infty(N,E))$ be a $C^\infty_{\mathbb{K}}$-map. We have to show that $\Psi \circ \gamma \colon I \to C^\infty(M \times N, E)$ is $C^\infty_{\mathbb{K}}$. By Prop. 12.1 (a), this will hold if we can show that

$$f := (\Psi \circ \gamma)^\wedge \colon I \times M \times N \to E$$

is of class $C^\infty_{\mathbb{K}}$ (since then $f^\vee = \Psi \circ \gamma$ apparently). This in turn holds if and only if f is a $c^\infty_{\mathbb{K}}$-map, the manifold $I \times M \times N$ being modeled on a metrizable topological vector space. However, given $\eta = (\eta_1, \eta_2, \eta_3) \colon \mathbb{K}^m \supseteq J \to I \times M \times N$ of class $C^\infty_{\mathbb{K}}$, we have

$$(f \circ \eta)(z) = \varepsilon_2\Big(\varepsilon_1\big(\gamma(\eta_1(z)), \eta_2(z)\big),\, \eta_3(z)\Big)\,,$$

where $\varepsilon_1 : C^\infty(M, C^\infty(N,E)) \times M \to C^\infty(N,E)$ and $\varepsilon_2 : C^\infty(N,E) \times N \to E$ are the respective evaluation maps, which are $c^\infty_{\mathbb{K}}$-maps by Lemma 12.5. Consequently, $f \circ \eta$ is of class $C^\infty_{\mathbb{K}}$ and thus f a $c^\infty_{\mathbb{K}}$-map, which completes the proof. $\square$

Note that, in the real or complex case, none of the topological vector spaces involved in Lemma 12.1, Proposition 12.2 and Proposition 12.6 need to be locally convex. Cf. [6] for a careful discussion of the exponential law for maps $M \times N \to E$, when E is a real locally convex space and both M and N are open subsets of real locally convex spaces. In the real and complex locally convex case, the exponential law for conveniently smooth maps plays a central role in convenient differential calculus (see [17], [47]). The reader should be aware that the locally convex topology on spaces of conveniently smooth functions primarily used in convenient differential calculus (initial with respect to pullbacks along smooth curves) is in general properly coarser than the topology we use here, already for $C^\infty(\mathbb{R}^2, \mathbb{R})$ (cf. [6]).

13 Diffeomorphism groups of finite-dimensional, paracompact smooth manifolds over local fields

Let $\mathbb{K}$ be a local field (of arbitrary characteristic), and M be a paracompact, finite-dimensional smooth manifold over $\mathbb{K}$. In this section, we turn the group $\mathrm{Diff}^\infty(M)$ of smooth diffeomorphisms of M into a $\mathbb{K}$-Lie group, modeled on the space $C^\infty_c(M, TM)$ of compactly supported smooth vector fields, equipped with the box topology.[16] Since $M = \coprod_{i \in I} B_i$ is a disjoint union of balls, we first turn the diffeomorphism group $\mathrm{Diff}^\infty(B)$

[16] By Prop. 8.13 (f), this is the locally convex direct limit topology on $C^\infty_c(M, TM) = \varinjlim_K C^\infty_K(M, TM)$.

of a ball B into a Lie group, which is quite easy. Then also the weak direct product $\prod^*_{i\in I} \mathrm{Diff}^\infty(B_i)$ is a Lie group by our general construction from Section 7. As this weak direct product can be identified with a subgroup of $\mathrm{Diff}^\infty(M)$ in an obvious way, it only remains to show in a final step that $\mathrm{Diff}^\infty(M)$ can be given a Lie group structure making $\prod^*_{i\in I} \mathrm{Diff}^\infty(B_i)$ an open subgroup; this will complete our construction.

In the next section, which can be read independently of the present one, we describe an alternative (slightly more complicated) construction, which is restricted to σ-compact manifolds but provides information also on the groups $\mathrm{Diff}^r(M)$ of C^r-diffeomorphisms for finite r.

13.1 Throughout this section and the next, $\mathbb{K}$ denotes a local field. We let $|.|$ be an ultrametric absolute value on $\mathbb{K}$ defining its topology, and $\mathbb{O}$ be the maximal compact subring of $\mathbb{K}$. Given $d \in \mathbb{N}$, we let $\|\bullet\|_\infty$ be the maximum norm on $\mathbb{K}^d$. Given $a \in \mathbb{K}^d$ and $\varepsilon > 0$, $B_\varepsilon(a) := \{y \in \mathbb{K}^d \colon \|y-a\|_\infty < \varepsilon\}$ denotes the open ball with respect to the maximum norm.

The diffeomorphism group of a ball

Given $d \in \mathbb{N}$, consider the ball $B := \mathbb{O}^d \subseteq \mathbb{K}^d$. We show:

Proposition 13.2 *The set* $\mathrm{Diff}^\infty(B)$ *of all* $C^\infty_{\mathbb{K}}$*-diffeomorphisms of* B *is an open subset of* $C^\infty(B,\mathbb{K}^d)$*. Consider* $\mathrm{Diff}^\infty(B)$ *as an open smooth submanifold of* $C^\infty(B,\mathbb{K}^d)$*. Then* $\mathrm{Diff}^\infty(B)$*, with composition of mappings as the group operation, is a* $\mathbb{K}$*-Lie group.*

Proof. We prove the proposition in various steps.

13.3 Define $\mathrm{End}^\infty(B) = C^\infty_B(B,B) = \{\gamma \in C^\infty(B,\mathbb{K}^d) \colon \gamma(B) \subseteq B\}$. Since B is both open and compact, Proposition 4.20 shows that $\mathrm{End}^\infty(B)$ is an open subset of $C^\infty(B,\mathbb{K}^d) = C^\infty_B(B,\mathbb{K}^d)$. By Proposition 11.2, the composition map

$$\Gamma \colon \mathrm{End}^\infty(B) \times \mathrm{End}^\infty(B) \to \mathrm{End}^\infty(B)\,, \quad \Gamma(\gamma,\eta) := \gamma \circ \eta$$

is of class $C^\infty_{\mathbb{K}}$. In particular, Γ is continuous and thus $\mathrm{End}^\infty(B)$ is a topological monoid, with identity element id_B. Hence, by standard arguments, the unit group $\mathrm{Diff}^\infty(B) = \mathrm{End}^\infty(B)^\times$ of the topological monoid $\mathrm{End}^\infty(B)$ will be open in $\mathrm{End}^\infty(B)$ (and hence in $C^\infty(B,\mathbb{K}^d)$) if we can show that it contains an identity neighbourhood.

13.4 Given $\gamma \in C^\infty(B,\mathbb{K}^d)$ and $x \in B$, we abbreviate $\gamma'(x) := d\gamma(x,\bullet)$. We let $\Omega \subseteq \mathrm{End}^\infty(B)$ be the set of all $\gamma \in \mathrm{End}^\infty(B)$ such that $(\gamma - \mathrm{id}_B)^{[1]}(B \times B \times \mathbb{O}) \subseteq B_{\frac{1}{2}}(0)$. Then Ω is an open identity neighbourhood in $\mathrm{End}^\infty(B)$ and $\|\gamma'(x) - \mathrm{id}\| < \frac{1}{2}$ for all $\gamma \in \Omega$ and $x \in B$ (using the operator norm with respect to the maximum norm $\|.\|_\infty$). We now show that $\Omega \subseteq \mathrm{Diff}^\infty(B)$. To this end, let $\gamma \in \Omega$ and abbreviate $\sigma := \gamma - \mathrm{id}_B$. Then $\gamma'(x) \in \mathrm{GL}_d(\mathbb{O}) = \mathrm{Iso}(\mathbb{K}^d, \|\bullet\|_\infty)$ is a linear isometry for all $x \in B$, because $\|\gamma'(x) - \mathrm{id}\| < \frac{1}{2}$

(cf. [74], Chapter IV, Appendix 1). Furthermore, $\|\gamma'(x)\| = \|\gamma'(x)^{-1}\| = 1$. We conclude that

$$\begin{aligned}\|\gamma(z)-\gamma(y)-\gamma'(x).(z-y)\|_\infty &= \|\sigma(z)-\sigma(y)-\sigma'(x).(z-y)\|_\infty\\ &\leq \min\{\,\|\sigma(z)-\sigma(y)\|_\infty,\ \|\sigma'(x).(z-y)\|_\infty\}\\ &< \frac{1}{2}\,\|z-y\|_\infty \;=\; \frac{1}{2\,\|\gamma'(x)^{-1}\|}\,\|z-y\|_\infty \qquad (36)\end{aligned}$$

for all $x,y,z\in B$ such that $y\neq z$. Indeed, because we are using the maximum norm here, given x,y,z as before there exists $0\neq\xi\in\mathbb{K}$ such that $|\xi|=\|z-y\|_\infty\leq 1$. Then $\|\sigma'(x).(z-y)\|_\infty\leq\|\sigma'(x)\|\,\|z-y\|_\infty<\frac{1}{2}\|z-y\|_\infty$ and

$$\sigma(z)-\sigma(y)=\xi\,\tfrac{1}{\xi}\big(\sigma\big(y+\xi\tfrac{z-y}{\xi}\big)-\sigma(y)\big)=\xi\,\sigma^{[1]}\big(y,\tfrac{z-y}{\xi},\xi\big)$$

with $\frac{z-y}{\xi}\in\mathbb{O}^d$ and $\xi\in\mathbb{O}$, entailing that $\|\sigma(z)-\sigma(y)\|_\infty\leq|\xi|\cdot\|\sigma^{[1]}(y,\frac{z-y}{\xi},\xi)\|_\infty<\frac{1}{2}|\xi|=\frac{1}{2}\|z-y\|_\infty$. Thus (36) holds. Using (36) with $x=0$, [28, Lemma 6.1 (b)] shows that γ is an isometry from B onto $\gamma(0)+\gamma'(0).B=\gamma(0)+B=B$. Since $\gamma'(x)\in\mathrm{GL}_d(\mathbb{O})$ is invertible for all x, we deduce from the Inverse Function Theorem [28, Thm. 7.3] that γ is a $C^\infty_{\mathbb{K}}$-diffeomorphism and thus $\gamma\in\mathrm{Diff}^\infty(B)$. We have shown that $\Omega\subseteq\mathrm{Diff}^\infty(B)$. Hence $\mathrm{Diff}^\infty(B)$ is open.

13.5 The group multiplication on $\mathrm{Diff}^\infty(B)$ being smooth by **13.3**, it only remains to show smoothness of the inversion map $\iota\colon\mathrm{Diff}^\infty(B)\to\mathrm{Diff}^\infty(B)$, $\iota(\gamma):=\gamma^{-1}$. We only need to prove that

$$\iota^\wedge\colon\mathrm{Diff}^\infty(B)\times B\to\mathbb{K}^d\,,\quad \iota^\wedge(\gamma,x):=\iota(\gamma)(x)=\gamma^{-1}(x)$$

is smooth; then $\iota=(\iota^\wedge)^\vee\colon\mathrm{Diff}^\infty(B)\to C^\infty(B,\mathbb{K}^d)$ will be smooth, by Lemma 12.1 (a). By Lemma 11.1, the evaluation map

$$\varepsilon\colon\mathrm{Diff}^\infty(B)\times B\to\mathbb{K}^d\,,\quad \varepsilon(\gamma,x):=\gamma(x)$$

is smooth. Note that $\varepsilon(\gamma,\bullet)=\gamma$ is a diffeomorphism of B for each $\gamma\in\mathrm{Diff}^\infty(B)$, where $\mathrm{Diff}^\infty(B)$ is an open subset of the metrizable topological vector space $C^\infty(B,\mathbb{K}^d)$ (see Proposition 4.19 (c)). Therefore the Inverse Function Theorem with Parameters [28, Thm. 8.1 (c)′] can be applied to the map ε, using the diffeomorphism $\gamma\in\mathrm{Diff}^\infty(B)$ itself as the parameter. The theorem shows that $\mathrm{Diff}^\infty(B)\times B\to\mathbb{K}^d$, $(\gamma,x)\mapsto(\varepsilon(\gamma,\bullet))^{-1}(x)=\gamma^{-1}(x)=\iota^\wedge(\gamma,x)$ is smooth. As just observed, this entails smoothness of ι. □

13.6 Slightly more generally, let us consider a $C^\infty_{\mathbb{K}}$-manifold M now which is isomorphic to $B=\mathbb{O}^d$ as a $C^\infty_{\mathbb{K}}$-manifold. Given a $C^\infty_{\mathbb{K}}$-diffeomorphism $\psi: M\to B$, we simply give $\mathrm{Diff}^\infty(M)$ the uniquely determined $\mathbb{K}$-Lie group structure modeled on $C^\infty(M,TM)\cong C^\infty(B,\mathbb{K}^d)$ which makes the isomorphism of groups

$$\Theta_\psi\colon\mathrm{Diff}^\infty(M)\to\mathrm{Diff}^\infty(B)\,,\quad \Theta_\psi(\gamma):=\psi\circ\gamma\circ\psi^{-1}$$

an isomorphism of Lie groups.

Lemma 13.7 *For $M \cong B$ as before, the Lie group structure on* $\mathrm{Diff}^\infty(M)$ *just defined is independent of the choice of $C^\infty_{\mathbb{K}}$-diffeomorphism $\psi\colon M \to B$.*

Proof. If both ϕ and ψ are $C^\infty_{\mathbb{K}}$-diffeomorphism $M \to B$, then the composition $\Theta_\phi \circ (\Theta_\psi)^{-1}\colon \mathrm{Diff}^\infty(B) \to \mathrm{Diff}^\infty(B)$, $\gamma \mapsto (\phi \circ \psi^{-1}) \circ \gamma \circ (\phi \circ \psi^{-1})^{-1}$ is an inner automorphism of the Lie group $\mathrm{Diff}^\infty(B)$ and hence a $C^\infty_{\mathbb{K}}$-diffeomorphism. The assertion follows. □

A very similar argument shows:

Lemma 13.8 *Suppose that M and N are finite-dimensional $C^\infty_{\mathbb{K}}$-manifolds such that $M \cong N \cong B$. Let $\phi\colon M \to N$ be a $C^\infty_{\mathbb{K}}$-diffeomorphism. Then*

$$\Lambda\colon \mathrm{Diff}^\infty(M) \to \mathrm{Diff}^\infty(N)\,, \qquad \gamma \mapsto \phi \circ \gamma \circ \phi^{-1}$$

is an isomorphism of Lie groups.

Proof. Let $\psi : N \to B$ be a $C^\infty_{\mathbb{K}}$-diffeomorphism. Then also $\psi \circ \phi : M \to B$ is a $C^\infty_{\mathbb{K}}$-diffeomorphism and hence $\Theta_\psi : \mathrm{Diff}^\infty(N) \to \mathrm{Diff}^\infty(B)$, $\gamma \mapsto \psi \circ \gamma \circ \psi^{-1}$ and $\Theta_{\psi\circ\phi} : \mathrm{Diff}^\infty(M) \to \mathrm{Diff}^\infty(B)$, $\gamma \mapsto (\psi \circ \phi) \circ \gamma \circ (\psi \circ \phi)^{-1}$ are isomorphisms of Lie groups. Hence also $\Lambda = \big(\Theta_\psi\big)^{-1} \circ \Theta_{\psi\circ\phi}$ is an isomorphism of Lie groups. □

Another technical lemma is useful:

Lemma 13.9 *If M is an open submanifold of $\mathbb{K}^d$ such that $M \cong B$, then* $\mathrm{Diff}^\infty(M)$ *is an open subset of $C^\infty(M, \mathbb{K}^d)$. The manifold structure making* $\mathrm{Diff}^\infty(M)$ *an open submanifold of $C^\infty(M, \mathbb{K}^d)$ coincides with the manifold structure underlying the Lie group* $\mathrm{Diff}^\infty(M)$*, as defined in* **13.6**.

Proof. Let $\psi : M \to B$ be a $C^\infty_{\mathbb{K}}$-diffeomorphism. As $M \subseteq \mathbb{K}^d$ is open and compact, $C^\infty(M, M)$ is open in $C^\infty(M, \mathbb{K}^d)$. The pullback $C^\infty(\psi^{-1}, \mathbb{K}^d)\colon C^\infty(M, \mathbb{K}^d) \to C^\infty(B, \mathbb{K}^d)$ is a linear isomorphism which takes $C^\infty(M, M)$ $C^\infty_{\mathbb{K}}$-diffeomorphically onto $C^\infty(B, M)$ (cf. Lemma 4.11). The map $C^\infty(B, \psi) : C^\infty(B, M) \to C^\infty(B, B)$ is a $C^\infty_{\mathbb{K}}$-diffeomorphism, since so is ψ (cf. Corollary 4.21). Hence also

$$\Phi\colon C^\infty(M, M) \to C^\infty(B, B), \quad \gamma \mapsto \psi \circ \gamma \circ \psi^{-1}$$

is a $C^\infty_{\mathbb{K}}$-diffeomorphism. Since $\mathrm{Diff}^\infty(B)$ is an open submanifold of $C^\infty(B, B)$, the set $\Phi^{-1}(\mathrm{Diff}^\infty(B)) = \mathrm{Diff}^\infty(M)$ is open in $C^\infty(M, M)$, and Φ induces a $C^\infty_{\mathbb{K}}$-diffeomorphism Θ_ψ from the open submanifold $\mathrm{Diff}^\infty(M) \subseteq C^\infty(M, M)$ onto $\mathrm{Diff}^\infty(B)$. But the same map Θ_ψ also is a $C^\infty_{\mathbb{K}}$-diffeomorphism from the Lie group $\mathrm{Diff}^\infty(M)$ onto $\mathrm{Diff}^\infty(B)$, by definition of the Lie group structure in **13.6**. □

Passage to paracompact manifolds

Let M be a paracompact, finite-dimensional smooth $\mathbb{K}$-manifold now, of dimension d, say. Then M is a disjoint union $M = \coprod_{i\in I} B_i$ of a family of open and compact balls $B_i \subseteq M$ (see Lemma 8.3 (b)). For each $i \in I$, we equip $\mathrm{Diff}^\infty(B_i)$ with the $\mathbb{K}$-Lie group structure modeled on $C^\infty(B_i, TB_i)$ described in **13.6** and Lemma 13.7. We then endow the weak direct product

$$\textstyle\prod^*_{i\in I} \mathrm{Diff}^\infty(B_i)$$

with a Lie group structure, as described in Proposition 7.1. Consider the mapping

$$\Psi : \textstyle\prod^*_{i\in I} \mathrm{Diff}^\infty(B_i) \to \mathrm{Diff}^\infty(M) \quad (\gamma_i)_{i\in I} \mapsto \displaystyle\coprod_{i\in I} \gamma_i \tag{37}$$

taking $(\gamma_i)_{i\in I}$ to the map $\gamma\colon M \to M$ determined by $\gamma|_{B_i} = \gamma_i$ for each $i \in I$. Then indeed Ψ takes its values in $\mathrm{Diff}^\infty(M)$, and apparently Ψ is injective and a homomorphism of groups. Throughout the following, using Ψ we shall identify $\prod^*_{i\in I} \mathrm{Diff}^\infty(B_i)$ with the corresponding subgroup $\mathrm{im}\,\Psi \subseteq \mathrm{Diff}^\infty(M)$. We shall also identify the modeling space $\bigoplus_{i\in I} C^\infty(B_i, TB_i)$ with $C^\infty_c(M, TM)$ in the obvious way (Proposition F.19 (e)); cf. Remark 8.17).

Lemma 13.10 *Assume that $M \cong B$ and assume that $M = \coprod_{j\in J} C_j$ for a finite family $(C_j)_{j\in J}$ of balls. Then $\prod_{j\in J} \mathrm{Diff}^\infty(C_j)$ is open in $\mathrm{Diff}^\infty(M)$ and $\mathrm{Diff}^\infty(M)$ induces the given manifold structure on the product $\prod_{j\in J} \mathrm{Diff}^\infty(C_j)$ of the Lie groups $\mathrm{Diff}^\infty(C_j)$.*

Proof. *Reduction to the case where M is a metric ball*: Let $\psi : M \to B$ be a $C^\infty_{\mathbb{K}}$-diffeomorphism. Then $\Theta_\psi\colon \mathrm{Diff}^\infty(M) \to \mathrm{Diff}^\infty(B)$, $\gamma \mapsto \psi \circ \gamma \circ \psi^{-1}$ is an isomorphism of Lie groups, by **13.6**. Set $B_j := \psi(C_j) \subseteq B$. Then $\Lambda_j\colon \mathrm{Diff}^\infty(C_j) \to \mathrm{Diff}^\infty(B_j)$, $\Lambda_j(\gamma) := \psi|^{B_j}_{C_j} \circ \gamma \circ \big(\psi|^{B_j}_{C_j}\big)^{-1}$ is an isomorphism of Lie groups for each $j \in J$, by Lemma 13.8. Since the restriction of Θ_ψ to $\prod_{j\in J} \mathrm{Diff}^\infty(C_j)$ coincides with the map $\prod_{j\in J} \Lambda_j$ onto $\prod_{j\in J} \mathrm{Diff}^\infty(B_j)$, we clearly only need to prove the assertion for $B = \coprod_{j\in J} B_j$ (then the assertion concerning $M = \coprod_{j\in J} C_j$ follows).

By the preceding, we may assume now that $M = B$. By Lemma 4.12, the map $\rho : C^\infty(M, \mathbb{K}^d) \to \prod_{j\in J} C^\infty(C_j, \mathbb{K}^d)$, $\rho(\gamma) := (\gamma|_{C_j})_{j\in J}$ is an isomorphism of topological vector spaces. By Lemma 13.9, the Lie group $\mathrm{Diff}^\infty(C_j)$ is an open submanifold of $C^\infty(C_j, \mathbb{K}^d)$. Hence ρ^{-1} induces an isomorphism Ψ from the direct product of Lie groups $P := \prod_{j\in J} \mathrm{Diff}^\infty(C_j)$ onto the open subset $\Psi(P) = \rho^{-1}(P) \subseteq \mathrm{Diff}^\infty(M)$ of $C^\infty(M, \mathbb{K}^d)$. Here Ψ is the map from (37). □

Lemma 13.11 *Assume that $M = \coprod_{i\in I} B_i$ is a disjoint union of a family $(B_i)_{i\in I}$ of balls $B_i \subseteq M$, and assume that, for each $i \in I$, the ball $B_i = \coprod_{j\in J_i} C_{ij}$ is a disjoint union of balls $C_{ij} \subseteq B_i$ for some finite set J_i. Abbreviate $K := \{(i,j) : i \in I, j \in J_i\}$. Then $\prod^*_{(i,j)\in K} \mathrm{Diff}^\infty(C_{ij})$ is an open subgroup of $\prod^*_{i\in I} \mathrm{Diff}^\infty(B_i)$, and $\prod^*_{i\in I} \mathrm{Diff}^\infty(B_i)$ induces the given manifold structure on the weak direct product $\prod^*_{(i,j)\in K} \mathrm{Diff}^\infty(C_{ij})$.*

Proof. By Lemma 7.2 (b), there is a natural isomorphism $\prod^*_{(i,j)\in K} \mathrm{Diff}^\infty(C_{ij}) \cong \prod^*_{i\in I} H_i$, with $H_i := \prod_{j\in J_i} \mathrm{Diff}^\infty(C_{ij})$. Here H_i is an open subgroup (and submanifold) of $\mathrm{Diff}^\infty(B_i)$, by Lemma 13.10. Hence, by Lemma 7.2 (a), also the weak direct product $\prod^*_{i\in I} H_i$ is an open subgroup and submanifold of $\prod_{i\in I} \mathrm{Diff}^\infty(B_i)$, as asserted. □

Theorem 13.12 (The Lie group structure on $\mathrm{Diff}^\infty(M)$) *Let M be a paracompact, finite-dimensional smooth manifold over a local field $\mathbb{K}$. Then there exists a uniquely determined $C^\infty_{\mathbb{K}}$-manifold structure on $\mathrm{Diff}^\infty(M)$, modeled on the space $C^\infty_c(M,TM)$ of compactly supported smooth vector fields, such that $\prod^*_{i\in I} \mathrm{Diff}^\infty(B_i)$ is an open subgroup of $\mathrm{Diff}^\infty(M)$ and $\mathrm{Diff}^\infty(M)$ induces the given manifold structure on $\prod^*_{i\in I} \mathrm{Diff}^\infty(B_i)$, for every cover $(B_i)_{i\in I}$ of M by mutually disjoint balls B_i.*

Proof. For the moment, we fix a cover $(B_i)_{i\in I}$ of M by mutually disjoint balls. Let $U := \prod^*_{i\in I} \mathrm{Diff}^\infty(B_i) \subseteq \mathrm{Diff}^\infty(M)$, equipped with its natural Lie group structure introduced above. Suppose we can show the following claim:

Claim. *For every $\gamma \in \mathrm{Diff}^\infty(M)$, there exists an open identity neighbourhood $V \subseteq U$ such that $I_\gamma(V) \subseteq U$, and such that $I_\gamma|_V^U : V \to U$ is smooth, where $I_\gamma : \mathrm{Diff}^\infty(M) \to \mathrm{Diff}^\infty(M)$, $I_\gamma(\eta) := \gamma \circ \eta \circ \gamma^{-1}$.*

Then there exists a uniquely determined Lie group structure on $\mathrm{Diff}^\infty(M)$ with U as an open submanifold, by Proposition 1.18.

13.13 To prove the claim, let $\gamma \in \mathrm{Diff}^\infty(M)$. As a consequence of Lemma 8.4, for each $i \in I$ there exists a finite cover $(C_{ij})_{j\in J_i}$ of B_i by mutually disjoint balls that is subordinate to the open cover $\{B_i \cap \gamma^{-1}(B_k) : k \in I\}$ of B_i. Let $J := \{(i,j) : i \in I, j \in J_i\}$. Given $k \in I$, define $L_k := \{(i,j) \in J : \gamma(C_{ij}) \subseteq B_k\}$. Let $L := \{(k,\ell) : k \in I, \ell \in L_k\}$ and $D_k := \gamma(C_\ell)$ for $(k,\ell) \in L$. Then $(D_{k\ell})_{\ell\in L_k}$ is a finite partition of B_k into balls, for each $k \in I$. The map $\pi : L \to J$, $\pi(k,\ell) := \ell$ is a bijection, and we have

$$\gamma(C_{\pi(k,\ell)}) = \gamma(C_\ell) = D_{k\ell} \quad \text{for all } (k,\ell) \in L. \tag{38}$$

Define $V := \prod^*_{(i,j)\in J} \mathrm{Diff}^\infty(C_{ij})$ and $W := \prod^*_{(k,\ell)\in L} \mathrm{Diff}^\infty(D_{k\ell})$; then V and W are open subgroups (and submanifolds) of U, by Lemma 13.11. Let $\eta \in V$. Given $x \in D_{k\ell}$, we have $\gamma^{-1}(x) \in C_\ell$ by (38) and hence $\eta(\gamma^{-1}(x)) \in C_\ell$. Thus $I_\gamma(\eta)(x) = \gamma(\eta(\gamma^{-1}(x))) \in D_{k\ell}$, using (38) again. Therefore $I_\gamma(V) \subseteq W \subseteq U$, as desired. Interpreting η as the corresponding element $(\eta_{ij})_{(i,j)\in J} \in \prod^*_{(i,j)\in J} \mathrm{Diff}^\infty(C_{ij})$ here with $\eta_{ij} := \eta|_{C_{ij}}$, by the preceding we have

$$I_\gamma(\eta)_{k\ell} := I_\gamma(\eta)|_{D_{k\ell}} \;=\; \Lambda_{k\ell}(\eta_{\pi(k,\ell)})$$

for all $(k,\ell) \in L$, where $\Lambda_{k\ell} : \mathrm{Diff}^\infty(C_{\pi(k,\ell)}) \to \mathrm{Diff}^\infty(D_{k\ell})$, $\Lambda_{k\ell}(\sigma) := \gamma|_{C_\ell} \circ \sigma \circ \gamma^{-1}|^{C_\ell}_{D_{k\ell}}$ is an isomorphism of $\mathbb{K}$-Lie groups by Lemma 13.8. Thus $I_\gamma|_V^W : V \to W$ is a mapping of the type discussed in Lemma 7.2 (c), and thus $I_\gamma|_V^W$ is an isomorphism of $\mathbb{K}$-Lie groups (and hence a $C^\infty_{\mathbb{K}}$-diffeomorphism). Thus, the above claim is established.

By the preceding, $\mathrm{Diff}^\infty(M)$ admits a unique $\mathbb{K}$-Lie group structure making the Lie group $\prod^*_{i\in I}\mathrm{Diff}^\infty(B_i)$ a subgroup and open submanifold. To complete the proof of Theorem 13.12, it only remains to show that the Lie group structure on $\mathrm{Diff}^\infty(M)$ so obtained is independent of the choice of the partition $M=\coprod_{i\in I}B_i$ of M into balls. To this end, suppose that $M=\coprod_{j\in J}C_j$ is a second partition of M into balls. For each $(i,j)\in I\times J$ such that $B_i\cap C_j\neq\emptyset$, the open, compact submanifold $C_i\cap C_j$ can be partitioned into finitely many balls (cf. Lemma 8.3 (b)). As a consequence, we find a partition $(D_k)_{k\in K}$ of M into balls that is subordinate to the disjoint open cover $\{B_i\cap C_j : i\in I, j\in J\}$ of M. Given $i\in I$, the set $K_i:=\{k\in K : D_k\subseteq B_i\}$ is finite, and $B_i=\coprod_{k\in K_i}D_k$. Likewise, for any $j\in J$ the set $L_j:=\{k\in K : D_k\subseteq C_j\}$ is finite, and $C_j=\coprod_{k\in L_j}D_k$. Hence, by Lemma 13.11, the weak direct product $\prod^*_{k\in K}\mathrm{Diff}^\infty(D_k)$ is a subgroup and open submanifold of both $U:=\prod^*_{i\in I}\mathrm{Diff}^\infty(B_i)$ and $V:=\prod^*_{j\in J}\mathrm{Diff}^\infty(C_j)$. Therefore $\prod^*_{k\in K}\mathrm{Diff}^\infty(D_k)$ is a subgroup and open submanifold of both $\mathrm{Diff}^\infty(M)$, equipped with the unique Lie group structure making U an open submanifold, and of $\mathrm{Diff}^\infty(M)$, equipped with the unique Lie group structure making V an open submanifold. As a consequence, the two Lie group structures on $\mathrm{Diff}^\infty(M)$ coincide. □

Remark 13.14 It is not hard to see that the natural action $\mathrm{Diff}^\infty(M)\times M\to M$ (the evaluation map) is smooth, entailing that every smooth homomorphism $\pi\colon G\to\mathrm{Diff}^\infty(M)$ from a $\mathbb{K}$-Lie group G to $\mathrm{Diff}^\infty(M)$ gives rise to a smooth action $\pi^\wedge\colon G\times M\to M$. If G is finite-dimensional here or modeled on a metrizable topological vector space, then G has an open subgroup fixing any element outside a compact subset of M. Conversely, given a smooth action $\sigma\colon G\times M\to M$, the associated homomorphism $\sigma^\vee\colon G\to\mathrm{Diff}^\infty(M)$ is smooth, provided there exists an open subgroup $U\subseteq G$ and a compact subset $K\subseteq M$ such that $\sigma(x,y)=y$ for all $x\in U$ and $y\in M\setminus K$. This condition is, of course, automatically satisfied if M is compact; in this special case, smooth actions of Lie groups on M are in one-to-one correspondence with smooth homomorphisms into $\mathrm{Diff}^\infty(M)$. Compare [33] for full proofs in the real case; they are easily adapted to the present situation.

In the real case, it is well known that every continuous action of a finite-dimensional Lie group on a manifold by C^∞-diffeomorphisms is automatically smooth [61, Thm., p. 212]. It is also known that every locally compact group acting effectively on a connected finite-dimensional smooth manifold by diffeomorphisms is a Lie group (see [45, Ch. I, Thm. 4.6] and [61, §5.2]), whence every locally compact subgroup of $\mathrm{Diff}^\infty(M)$ is a Lie group in particular. It is natural to ask for analogues in the p-adic case. The following problems are open and deserve to be investigated:

Problem 13.15 Is every compact subgroup G of the diffeomorphism group $\mathrm{Diff}^\infty(M)$ of a paracompact, finite-dimensional smooth p-adic manifold M a p-adic Lie group? Does this hold at least if G is topologically finitely generated?

One strategy might be to try to verify the hypotheses of Lazard's characterization of finite-dimensional p-adic Lie groups (see [48, A1, Thm. (1.9)], [15, Thm. 8.32]; cf. [74, p. 157]).

However, the author suspects that counterexamples can be found. Compare also [55] for some related studies of subgroups of diffeomorphism groups.

Problem 13.16 Are continuous actions of finite-dimensional p-adic Lie groups by smooth diffeomorphisms on paracompact, finite-dimensional smooth p-adic manifolds always smooth? Arguing as in the real case (see [58] or [33]), this would entail that every continuous homomorphism from a finite-dimensional p-adic Lie group to $\mathrm{Diff}^\infty(M)$ is smooth.

14 The diffeomorphism groups $\mathrm{Diff}^r(M)$ and $\mathrm{Diff}^\infty(M)\breve{}$

Let $\mathbb{K}$ be a totally disconnected local field and M be a σ-compact smooth manifold over $\mathbb{K}$, of positive finite dimension d. Given $r \in \mathbb{N} \cup \{\infty\}$, let $\mathrm{Diff}^r(M)$ be the group of all $C^r_{\mathbb{K}}$-diffeomorphisms of M. In this section, we equip $\mathrm{Diff}^r(M)$ with a $C^\infty_{\mathbb{K}}$-manifold structure modeled on the space $C^r_c(M,TM)$ of compactly supported vector fields of class $C^r_{\mathbb{K}}$ on M which makes $\mathrm{Diff}^r(M)$ a topological group, with smooth right translation maps. For $r = \infty$, the preceding smooth manifold structure makes $\mathrm{Diff}^\infty(M)$ a Lie group, modeled on the LF-space $C^\infty_c(M,TM)$; it coincides with the Lie group constructed in the preceding section. However, we shall also define a *second* smooth manifold structure on $\mathrm{Diff}^\infty(M)$ making it a $\mathbb{K}$-Lie group, denoted $\mathrm{Diff}^\infty(M)\breve{}$; it is modeled on the projective limit

$$C^\infty_c(M,TM)\breve{} := \lim_{\longleftarrow k\in\mathbb{N}_0} C^k_c(M,TM) = \bigcap_{k\in\mathbb{N}_0} C^k_c(M,TM)$$

of LB-spaces. Note that $C^\infty_c(M,TM)\breve{}$ coincides with $C^\infty_c(M,TM)$ as a vector space, but its topology is coarser (and can be properly coarser). Since M is diffeomorphic to an open subset U of $\mathbb{K}^d$ (see Lemma 8.3 (a)), we first discuss $\mathrm{Diff}^r(U)$ and only pass to general M at the very end. Throughout this section, we retain the conventions from **13.1**.

The monoids $\mathrm{End}^r_c(U)$ and $\mathrm{End}^\infty_c(U)\breve{}$

14.1 Let $d \in \mathbb{N}$ and $U \subseteq \mathbb{K}^d$ be a non-empty, open subset. By Lemma 8.4, there exist a countable set I, positive real numbers r_i for $i \in I$ and elements $a_i \in U$ such that $U = \bigcup_{i\in I} B_{r_i}(a_i)$ as a disjoint union. Abbreviate $U_i := B_{r_i}(a_i)$. Then, for every $r \in \mathbb{N}_0 \cup \{\infty\}$, by Proposition 8.13 (e) the map

$$C^r_c(U,\mathbb{K}^d) \to \bigoplus_{i\in I} C^r(U_i,\mathbb{K}^d), \quad \gamma \mapsto (\gamma|_{U_i})_{i\in I}$$

is an isomorphism of topological $\mathbb{K}$-vector spaces (when the box topology is used on the direct sum); we use it to identify $C^r_c(U,\mathbb{K}^d)$ and $\bigoplus_{i\in I} C^r(U_i,\mathbb{K}^d)$. We define

$$\mathrm{End}^r_c(U) := \{\gamma \in C^r(U,\mathbb{K}^d) \colon\ \gamma(U) \subseteq U \text{ and } \gamma - \mathrm{id}_U \in C^r_c(U,\mathbb{K}^d)\} \quad \text{and}$$

$$\mathcal{E}^r_c(U) := \{\gamma \in C^r_c(U,\mathbb{K}^d) \colon \mathrm{id}_U + \gamma \in \mathrm{End}^r_c(U)\}\,.$$

Then

$$\beta_r \colon \mathcal{E}^r_c(U) \to \mathrm{End}^r_c(U), \quad \beta_r(\gamma) := \mathrm{id}_U + \gamma$$

is a bijection.

14.2 Every $\gamma \in \mathrm{End}^r_c(U)$ is a proper map, since $\gamma^{-1}(K) \subseteq K \cup \mathrm{supp}(\gamma - \mathrm{id}_U)$ for every compact subset K of U. Given $\gamma, \eta \in \mathcal{E}^r_c(U)$, we have

$$(\mathrm{id}_U + \gamma) \circ (\mathrm{id}_U + \eta) \; = \; \mathrm{id}_U + \eta + \gamma \circ (\mathrm{id}_U + \eta)\,,$$

where $\gamma \circ (\mathrm{id}_U + \eta) \in C^r_c(U, \mathbb{K}^d)$ since $\mathrm{id}_U + \eta$ is proper, and thus $\eta + \gamma \circ (\mathrm{id}_U + \eta) \in C^r_c(U, \mathbb{K}^d)$. Therefore $(\mathrm{id}_U + \gamma) \circ (\mathrm{id}_U + \eta) \in \mathrm{End}^r_c(U)$. We have shown that $\mathrm{End}^r_c(U)$ is closed under composition of maps, and thus $\mathrm{End}^r_c(U)$ is a monoid under composition, with id_U as the identity element. We give $\mathcal{E}^r_c(U)$ the monoid structure which makes β_r an isomorphism of monoids. Thus 0 is the identity in $\mathcal{E}^r_c(U)$, and the monoid multiplication $\mu_r \colon \mathcal{E}^r_c(U) \times \mathcal{E}^r_c(U) \to \mathcal{E}^r_c(U)$ is given by

$$\mu_r(\gamma, \eta) = \eta + \gamma \circ (\mathrm{id}_U + \eta)\,. \tag{39}$$

14.3 We claim that $\mathcal{E}^r_c(U)$ is open in $C^r_c(U, \mathbb{K}^d)$. In fact, let $\gamma \in \mathcal{E}^r_c(U)$ be given. For every $x \in U$, there exists $i(x) \in I$ such that $x + \gamma(x) \in U_{i(x)} = B_{r_{i(x)}}(a_{i(x)}) = x + \gamma(x) + B_{r_{i(x)}}(0) \subseteq U$. There exists $s(x) \in \,]0, r_{i(x)}]$ such that $B_{s(x)}(x) \subseteq U$ and $\gamma(B_{s(x)}(x)) \subseteq \gamma(x) + B_{r_{i(x)}}(0)$. By Lemma 8.4, there exists a countable cover $(V_j)_{j \in J}$ of U by mutually disjoint compact open sets V_j, which is subordinate to $(B_{s(x)}(x))_{x \in U}$. Given $j \in J$, choose $x_j \in U$ such that $V_j \subseteq B_{s(x_j)}(x_j)$, and abbreviate $i(j) := i(x_j)$. If $\eta \in C^r_c(U, \mathbb{K}^d)$ such that $\eta(V_j) \subseteq B_{r_{i(j)}}(\gamma(x_j))$, then

$$y + \eta(y) \in x_j + B_{s(x_j)}(0) + \gamma(x_j) + B_{r_{i(j)}}(0) = x_j + \gamma(x_j) + B_{r_{i(j)}}(0) = U_{i(j)} \subseteq U \tag{40}$$

for all $y \in V_j$. We have shown that the open neighbourhood $\bigoplus_{j \in J} C^r(V_j, B_{r_{i(j)}}(\gamma(x_j)))$ of γ in $C^r_c(U, \mathbb{K}^d)$ is contained in $\mathcal{E}^r_c(U)$. Thus $\mathcal{E}^r_c(U)$ is indeed open in $C^r_c(U, \mathbb{K}^d)$.

14.4 We consider $\mathcal{E}^r_c(U)$ as an open $C^\infty_{\mathbb{K}}$-submanifold of $C^r_c(U, \mathbb{K}^d)$. We equip $\mathrm{End}^r_c(U)$ with the smooth $\mathbb{K}$-manifold structure making $\beta_r \colon \mathcal{E}^r_c(U) \to \mathrm{End}^r_c(U)$ a $C^\infty_{\mathbb{K}}$-diffeomorphism.

14.5 Define $C^\infty_c(U, \mathbb{K}^d)\breve{} := \bigcap_{k \in \mathbb{N}_0} C^k_c(U, \mathbb{K}^d) = \varprojlim_{k \in \mathbb{N}_0} C^k_c(U, \mathbb{K}^d)$. Then the vector space underlying $C^\infty_c(U, \mathbb{K}^d)\breve{}$ is $C^\infty_c(U, \mathbb{K}^d)$, but the projective limit topology on $C^\infty_c(U, \mathbb{K}^d)\breve{}$ can be properly coarser than the topology on $C^\infty_c(U, \mathbb{K}^d)$ if U is non-compact. Since

$$\mathcal{E}^r_c(U) = \mathcal{E}^0_c(U) \cap C^r(U, \mathbb{K}^d) \tag{41}$$

for each $r \in \mathbb{N}_0 \cup \{\infty\}$, we have $\mathcal{E}^\infty_c(U) = \mathcal{E}^0_c(U) \cap C^\infty_c(U, \mathbb{K}^d)$ in particular. As a consequence, $\mathcal{E}^\infty_c(U)$ is an open subset of $C^\infty_c(U, \mathbb{K}^d)\breve{}$. When equipped with the topology induced by $C^\infty_c(U, \mathbb{K}^d)\breve{}$, we write $\mathcal{E}^\infty_c(U)\breve{}$ for $\mathcal{E}^\infty_c(U)$. We write $\mathrm{End}^\infty_c(U)\breve{}$ for the monoid $\mathrm{End}^\infty_c(U)$, equipped with the $C^\infty_{\mathbb{K}}$-manifold structure making $\tilde{\beta} \colon \mathcal{E}^\infty_c(U)\breve{} \to \mathrm{End}^\infty_c(U)\breve{}$, $\gamma \mapsto \mathrm{id}_U + \gamma$ a $C^\infty_{\mathbb{K}}$-diffeomorphism and an isomorphism of monoids.

In various steps, we now show:

Proposition 14.6 *In the preceding situation, we have:*

(a) *For every $r, k \in \mathbb{N}_0 \cup \{\infty\}$, the mapping*

$$m_{r,k}\colon \mathrm{End}_c^{r+k}(U) \times \mathrm{End}_c^r(U) \to \mathrm{End}_c^r(U), \quad m_{r,k}(\gamma, \eta) := \gamma \circ \eta$$

is of class $C^k_{\mathbb{K}}$. In particular, for each $r \in \mathbb{N}_0$ the composition map $m_r := m_{r,0}$: $\mathrm{End}_c^r(U) \times \mathrm{End}_c^r(U) \to \mathrm{End}_c^r(U)$ is continuous, and the composition maps $m_\infty := m_{\infty,\infty}\colon \mathrm{End}_c^\infty(U) \times \mathrm{End}_c^\infty(U) \to \mathrm{End}_c^\infty(U)$ and

$$\widetilde{m}\colon \mathrm{End}_c^\infty(U)\breve{} \times \mathrm{End}_c^\infty(U)\breve{} \to \mathrm{End}_c^\infty(U)\breve{}, \quad \widetilde{m}(\gamma, \eta) := \gamma \circ \eta$$

are smooth.

(b) *For every $r \in \mathbb{N}_0 \cup \{\infty\}$ and $\eta \in \mathcal{E}_c^r(U)$, the right translation map $\rho_{r,\eta} := m_{r,0}(\bullet, \eta)$: $\mathrm{End}_c^r(U) \to \mathrm{End}_c^r(U)$ is of class $C^\infty_{\mathbb{K}}$.*

(c) *For every $r \in \mathbb{N} \cup \{\infty\}$, the group of invertible elements $\mathrm{Diff}_c^r(U) := \mathrm{End}_c^r(U)^\times$ is open in $\mathrm{End}_c^r(U)$, and $\mathrm{Diff}_c^r(U) = \mathrm{Diff}_c^1(U) \cap \mathrm{End}_c^r(U)$. Also $\mathrm{Diff}_c^\infty(U)\breve{} := (\mathrm{End}_c^\infty(U)\breve{})^\times$ is open in $\mathrm{End}_c^\infty(U)\breve{}$.*

(d) *Given $r \in \mathbb{N} \cup \{\infty\}$, let γ be a $C^r_{\mathbb{K}}$-diffeomorphism of U. Then $\gamma \in \mathrm{Diff}_c^r(U)$ if and only if there exists a compact subset $K \subseteq U$ such that $\gamma(x) = x$ for all $x \in X \setminus K$.*

(e) *For each $r \in \mathbb{N} \cup \{\infty\}$ and $k \in \mathbb{N}_0 \cup \{\infty\}$, the map $\iota_{r,k} : \mathrm{Diff}_c^{r+k}(U) \to \mathrm{Diff}_c^r(U)$, $\gamma \mapsto \gamma^{-1}$ is $C^k_{\mathbb{K}}$. In particular, inversion $\iota_r := \iota_{r,0}\colon \mathrm{Diff}_c^r(U) \to \mathrm{Diff}_c^r(U)$ is continuous for each $r \in \mathbb{N}$, and the inversion maps $\iota_\infty := \iota_{\infty,\infty} : \mathrm{Diff}_c^\infty(U) \to \mathrm{Diff}_c^\infty(U)$ and $\tilde{\iota}\colon \mathrm{Diff}_c^\infty(U)\breve{} \to \mathrm{Diff}_c^\infty(U)\breve{}$ are smooth.*

Thus $\mathrm{Diff}_c^\infty(U)$ and $\mathrm{Diff}_c^\infty(U)\breve{}$ are $\mathbb{K}$-Lie groups when considered as open $C^\infty_{\mathbb{K}}$- submanifolds of $\mathrm{End}_c^\infty(U)$, resp., $\mathrm{End}_c^\infty(U)\breve{}$. Furthermore, $\mathrm{Diff}_c^r(U)$ is a topological group with respect to the topology underlying the smooth manifold $\mathrm{Diff}_c^r(U)$, for each $r \in \mathbb{N}$.

Proof. We begin with the proof of (a) and (b).

14.7 The maps β_r and β_{r+k} being $C^\infty_{\mathbb{K}}$-diffeomorphisms and isomorphism of monoids, in view of (39) apparently $m_{r,k}$ will be of class $C^k_{\mathbb{K}}$ if we can show that the mapping $\mu_{r,k}$: $\mathcal{E}_c^{r+k}(U) \times \mathcal{E}_c^r(U) \to C_c^r(U, \mathbb{K}^d)$, $(\gamma, \eta) \mapsto \eta + \gamma \circ (\mathrm{id}_U + \eta)$ is of class $C^k_{\mathbb{K}}$. The map $(\gamma, \eta) \mapsto \eta$ involved being continuous linear and thus smooth, it suffices to show that

$$f\colon C_c^{r+k}(U, \mathbb{K}^d) \times \mathcal{E}_c^r(U) \to C_c^r(U, \mathbb{K}^d), \quad f(\gamma, \eta) := \gamma \circ (\mathrm{id}_U + \eta)$$

is of class $C^k_{\mathbb{K}}$. To see this, let $\gamma \in C_c^{r+k}(U, \mathbb{K}^d)$, $\eta \in \mathcal{E}_c^r(U)$ be given. As in **14.3**, we find a countable open cover $(V_j)_{j \in J}$ of U by mutually disjoint compact open sets V_j, elements $x_j \in U$, and a mapping $i : J \to I$ such that $\eta \in \bigoplus_{j \in J} C^r(V_j, B_{r_{i(j)}}(\eta(x_j))) \subseteq \mathcal{E}_c^r(U)$, and such that $(\mathrm{id}_U + \tau)(V_j) \subseteq U_{i(j)}$ for all $j \in J$ and $\tau \in \bigoplus_{j \in J} C^r\big(V_j, B_{r_{i(j)}}(\eta(x_j))\big)$ (cf.

(40)). Abbreviate $B_j := B_{r_{i(j)}}(\eta(x_j))$ and $Q := \bigoplus_{j\in J} C^r(V_j, B_j)$. Then $C^{r+k}_c(U,\mathbb{K}^d)\times Q$ is an open neighbourhood of (γ,η). By the preceding, $f(\sigma,\tau)|_{V_j} = \sigma\circ(\mathrm{id}_U+\tau)|_{V_j} = \sigma|_{U_{i(j)}}\circ(\mathrm{id}_{V_j}+\tau|_{V_j})|^{U_{i(j)}}$ for all $\sigma\in C^{r+k}_c(U,\mathbb{K}^d)$ and $\tau\in Q$. Thus $f|_{C^{r+k}_c(U,\mathbb{K}^d)\times Q}$ can be written as the composition

$$\begin{aligned}
C^{r+k}_c(U,\mathbb{K}^d)\times Q \quad &\xrightarrow{\cong}\quad (\textstyle\bigoplus_{i\in I} C^{r+k}(U_i,\mathbb{K}^d))\times(\bigoplus_{j\in J} C^r(V_j,B_j))\\
&\xrightarrow{\cong}\quad \textstyle\bigoplus_{i\in I,j\in J}(C^{r+k}(U_i,\mathbb{K}^d)\times C^r(V_j,B_j))\\
&\xrightarrow{p}\quad \textstyle\bigoplus_{j\in J}(C^{r+k}(U_{i(j)},\mathbb{K}^d)\times C^r(V_j,B_j))\\
&\xrightarrow{\oplus_{j\in J} f_j}\quad \textstyle\bigoplus_{j\in I} C^r(V_j,\mathbb{K}^d)\xrightarrow{\cong} C^r_c(U,\mathbb{K}^d)
\end{aligned}$$

where "$\cong$" are the obvious isomorphisms of topological vector spaces (or their restrictions to $C^\infty_{\mathbb{K}}$-diffeomorphisms of open sets), p is the map $(\sigma_i,\tau_j)_{i\in I,j\in J}\mapsto(\sigma_{i(j)},\tau_j)_{j\in J}$ which is $C^\infty_{\mathbb{K}}$ as the restriction of a continuous linear map, and $f_j\colon C^{r+k}(U_{i(j)},\mathbb{K}^d)\times C^r(V_j,B_j)\to C^r(V_j,\mathbb{K}^d)$, $f_j(\sigma,\tau) := \sigma\circ(\mathrm{id}_{V_j}+\tau)|^{U_{i(j)}}$. Then $f_j = \Gamma_j\circ(\mathrm{id}_{C^{r+k}(U_{i(j)},\mathbb{K}^d)}\times g_j)$, where the composition map

$$\Gamma_j\colon C^{r+k}(U_{i(j)},\mathbb{K}^d)\times C^r(V_j,U_{i(j)})\to C^r(V_j,\mathbb{K}^d)$$

is $C^k_{\mathbb{K}}$ by Proposition 11.4,[17] and $g_j : C^r(V_j,B_j)\to C^r(V_j,U_{i(j)})$, $g_j(\tau) := \mathrm{id}_{V_j}+\tau$ is smooth, being a restriction of an affine-linear map. Thus each f_j is of class $C^k_{\mathbb{K}}$ and hence so is $\oplus_{j\in J} f_j$, by Proposition 6.9. Being a composition of $C^k_{\mathbb{K}}$-maps, $f|_{C^{r+k}_c(U,\mathbb{K}^d)\times Q}$ is $C^k_{\mathbb{K}}$.

If $k=0$ here, then $f({\scriptstyle\bullet},\eta)\colon C^r_c(U,\mathbb{K}^d)\to C^r_c(U,\mathbb{K}^d)$ is a continuous map by the preceding. Since it is also linear, we deduce that the map $f({\scriptstyle\bullet},\eta)$ is smooth. As a consequence, $\rho_{r,\eta} = m_{r,0}({\scriptstyle\bullet},\eta)$ is smooth, establishing (b).

14.8 Clearly $\widetilde{m}$ will be smooth if we can show that $\widetilde{\mu}\colon \mathcal{E}^\infty_c(U)\tilde{}\times\mathcal{E}^\infty_c(U)\tilde{}\to C^\infty_c(U,\mathbb{K}^d)\tilde{}$, $\widetilde{\mu}(\gamma,\eta) := \eta+\gamma\circ(\mathrm{id}_U+\eta)$ is smooth. By Lemma 1.17, $\widetilde{\mu}$ will be smooth if $\lambda_r\circ\widetilde{\mu}$ is smooth for every $r\in\mathbb{N}_0$, where $\lambda_r\colon C^\infty_c(U,\mathbb{K}^d)\tilde{}\to C^r_c(U,\mathbb{K}^d)$ is the inclusion map. But this is the case. In fact, given any $k\in\mathbb{N}_0$, we have $(\lambda_r\circ\widetilde{\mu})|^{\mathcal{E}^r_c(U)} = \mu_{r,k}\circ(\lambda_{r+k}\times\lambda_r)|^{\mathcal{E}^{r+k}_c(U)\times\mathcal{E}^r_c(U)}_{(\mathcal{E}^\infty_c(U)\tilde{})^2}$, where $\mu_{r,k}$ is of class $C^k_{\mathbb{K}}$ and λ_{r+k} and λ_r are continuous linear and thus smooth. Thus $\lambda_r\circ\widetilde{\mu}$ is of class $C^k_{\mathbb{K}}$. This completes the proof of Part (a) of Proposition 14.6.

14.9 To prove (d), let $r\in\mathbb{N}\cup\{\infty\}$. If $\gamma\in\mathrm{Diff}^r_c(U)$, then $\gamma\in\mathrm{End}^r_c(U)$, whence there exists a compact subset $K\subseteq U$ such that $(\gamma-\mathrm{id}_U)|_{U\setminus K}=0$ and thus $\gamma|_{U\setminus K} = \mathrm{id}_U|_{U\setminus K}$. Conversely, if γ is a $C^r_{\mathbb{K}}$-diffeomorphism of X such that γ coincides with id_U off some compact set K, then apparently also the inverse map γ^{-1} satisfies $\gamma^{-1}|_{X\setminus K} = \mathrm{id}_U|_{X\setminus K}$, whence $\gamma^{-1}\in\mathrm{End}^r_c(U)$. Thus γ is invertible in the monoid $\mathrm{End}^r_c(U)$; (c) is established.

14.10 To prove (c), note first that

$$\mathrm{End}^r_c(U)\cap\mathrm{End}^1_c(U)^\times = \mathrm{End}^r_c(U)^\times \quad\text{for all } r\in\mathbb{N}\cup\{\infty\}. \tag{42}$$

[17]We apply the proposition with $K := Y := V_j$; note that $C^r(V_j,U_{i(j)}) = \lfloor V_j,U_{i(j)}\rfloor_r$ here.

In fact, clearly $\mathrm{End}^r_c(U)^\times \subseteq \mathrm{End}^r_c(U)\cap\mathrm{End}^1_c(U)^\times$. If, conversely, $\gamma \in \mathrm{End}^r_c(U)\cap\mathrm{End}^1_c(U)^\times$, then γ is a $C^1_{\mathbb{K}}$-diffeomorphism and thus $d\gamma(x,\bullet)$ is invertible for all $x \in U$. Since, furthermore, γ is of class $C^r_{\mathbb{K}}$, the Ultrametric Inverse Function Theorem [28, Thm. 7.3] entails that γ^{-1} is of class $C^r_{\mathbb{K}}$ (cf. also [28, Rem. 5.4]). Thus γ is a $C^r_{\mathbb{K}}$-diffeomorphism, and, in view of (d), apparently $\gamma \in \mathrm{Diff}^r_c(U) = \mathrm{End}^r_c(U)^\times$.

14.11 Given $r \in \mathbb{N}\cup\{\infty\}$, by the preceding $\mathrm{End}^r_c(U)$ is a topological monoid. Therefore its unit group $\mathrm{End}^r_c(U)^\times$ will be open if we can show that it is an identity neighbourhood. The inclusion map $\mathrm{End}^r_c(U) \to \mathrm{End}^1_c(U)$ being continuous, in view of (42), we only need to show that $\mathrm{Diff}^1_c(U)$ is open in $\mathrm{End}^1_c(U)$, or equivalently, that $\mathcal{E}^1_c(U)^\times$ is a 0-neighbourhood in $\mathcal{E}^1_c(U)$. Let $B_i := B_{r_i}(0) \subseteq \mathbb{K}^d$ and $D_i := \{t \in \mathbb{K}\colon |t| < r_i\}$ for $i \in I$, with r_i as in **14.1**. Then $W := \bigoplus_{i\in I} C^1(U_i, B_i)$ is an open zero-neighbourhood in $C^1_c(U,\mathbb{K}^d)$, and we have $W \subseteq \mathcal{E}_c(U)$ since $x+\gamma(x) \in U_i + B_i = a_i + B_{r_i}(0) + B_{r_i}(0) = a_i + B_{r_i}(0) = U_i \subseteq U$ for all $\gamma \in W$, $i \in I$, and $x \in U_i$. Define

$$\Omega_i := \{\sigma \in C^1(U_i,B_i)\colon \sigma^{[1]}(U_i\times\mathbb{O}^d\times D_i) \subseteq B_{\frac{1}{2}}(0) \text{ and } d\sigma(U_i\times\mathbb{O}^d) \subseteq B_{\frac{1}{2}}(0)\}\,.$$

Then $\Omega := \bigoplus_{i\in I}\Omega_i \subseteq W \subseteq \mathcal{E}^1_c(U)$ is an open zero-neighbourhood. We claim that $\Omega \subseteq \mathcal{E}^1_c(U)^\times$, or equivalently, $\beta_1(\Omega) \subseteq \mathrm{Diff}^1_c(U)$. To see this, let $\sigma = (\sigma_i)_{i\in I} \in \Omega$, where $\sigma_i = \sigma|_{U_i} \in \Omega_i$ for $i \in I$. Define $\gamma := \beta_1(\sigma) = \mathrm{id}_U + \sigma$ and $\gamma_i = \mathrm{id}_{U_i} + \sigma_i = \gamma|_{U_i}$. Then $\gamma_i'(x) := d\gamma_i(x,\bullet) = \mathbf{1} + d\sigma_i(x,\bullet) \in \mathrm{GL}_d(\mathbb{O}) = \mathrm{Iso}(\mathbb{K}^d,\|\bullet\|_\infty)$ for all $x \in U_i$ (cf. [74], Chapter IV, Appendix 1) and $\|\gamma_i'(x)\| = \|\gamma_i'(x)^{-1}\| = 1$, because $\|\gamma_i'(x)-\mathbf{1}\| = \|\sigma_i'(x)\| < \frac{1}{2}$. We conclude that

$$\begin{aligned}\|\gamma_i(z)-\gamma_i(y)-\gamma_i'(x).(z-y)\|_\infty &= \|\sigma_i(z)-\sigma_i(y)-\sigma_i'(x).(z-y)\|_\infty\\ &\le \min\{\|\sigma_i(z)-\sigma_i(y)\|_\infty,\, \|\sigma_i'(x).(z-y)\|_\infty\}\\ &< \frac{1}{2}\|z-y\|_\infty \;=\; \frac{1}{2\,\|\gamma_i'(x)^{-1}\|}\,\|z-y\|_\infty \qquad (43)\end{aligned}$$

for all $x,y,z \in U_i$ such that $y \neq z$. Indeed, because we are using the supremum norm here, given x,y,z as before there exists $0 \neq \xi \in \mathbb{K}$ such that $|\xi| = \|z-y\|_\infty < r_i$. Then $\|\sigma_i'(x).(z-y)\|_\infty \le \|\sigma_i'(x)\|\,\|z-y\|_\infty < \frac{1}{2}\|z-y\|_\infty$ and

$$\sigma_i(z)-\sigma_i(y) = \xi\,\tfrac{1}{\xi}\big(\sigma_i(y+\xi\tfrac{z-y}{\xi})-\sigma_i(y)\big) = \xi\,\sigma_i^{[1]}(y,\tfrac{z-y}{\xi},\xi)$$

with $\frac{z-y}{\xi} \in \mathbb{O}^d$ and $\xi \in D_i$, entailing that $\|\sigma_i(z)-\sigma_i(y)\|_\infty \le |\xi|\cdot\|\sigma_i^{[1]}(y,\frac{z-y}{\xi},\xi)\|_\infty < \frac{1}{2}|\xi| = \frac{1}{2}\|z-y\|_\infty$. Thus (43) holds. Using (43) with $x = a_i$, [28, Lemma 6.1 (b)] shows that γ_i is an isometry from $U_i = B_{r_i}(a_i)$ onto $\gamma_i(a_i)+\gamma_i'(a_i).B_{r_i}(0) = \gamma_i(a_i)+B_{r_i}(0) = a_i+\sigma_i(a_i)+B_{r_i}(0) = a_i+B_{r_i}(0) = B_{r_i}(a_i) = U_i$. As a consequence, γ is an isometry from U onto U. Since $\gamma'(x) = \mathbf{1}+\sigma'(x) \in \mathrm{GL}_d(\mathbb{O})$ is invertible for all x, we deduce from the Inverse Function Theorem [28, Thm. 7.3] that γ is a $C^1_{\mathbb{K}}$-diffeomorphism and thus $\gamma \in \mathrm{Diff}^1_c(U)$, using (d).

14.12 In view of (42), the openness of $\mathrm{End}^1_c(U)^\times$ in $\mathrm{End}^1_c(U)$ just established entails that $\mathrm{Diff}^\infty_c(U)\breve{} = \mathrm{End}^\infty_c(U)\breve{}\cap \mathrm{End}^1_c(U)^\times$ is open in $\mathrm{End}^\infty_c(U)\breve{}$. This completes the proof of Part (c) of Proposition 14.6.

14.13 To prove (e), we first observe that, for given $r \in \mathbb{N}\cup\{\infty\}$ and $k \in \mathbb{N}_0\cup\{\infty\}$, the map $\iota_{r,k}$ will be of class $C^k_{\mathbb{K}}$ if its restriction to some open identity neighbourhood $Y \subseteq \mathrm{Diff}^{r+k}_c(U)$ is of class $C^k_{\mathbb{K}}$. In fact, if $\gamma \in \mathrm{Diff}^{r+k}_c(U)$ is given, then $Y\circ\gamma$ is an open neighbourhood of γ in $\mathrm{Diff}^{r+k}_c(U)$ since right translation by γ is a $C^\infty_{\mathbb{K}}$-diffeomorphism of $\mathrm{End}^{r+k}_c(U)$, as a consequence of (b). In view of (a) and (b), the formula $\iota_{r,k}|_{Y\circ\gamma}(\eta) = \eta^{-1} = \gamma^{-1}\circ(\eta\circ\gamma^{-1})^{-1} = m_{r,k}\Big(\gamma^{-1}, \iota_{r,k}|_Y(\rho_{r+k,\gamma^{-1}}(\eta))\Big)$ for $\eta \in Y\circ\gamma$ shows that $\iota_{r,k}$ will be $C^k_{\mathbb{K}}$ on $Y\circ\gamma$ if it is $C^k_{\mathbb{K}}$ on Y.

14.14 By **14.4** and the preceding, $\iota_{r,k}$ will be of class $C^k_{\mathbb{K}}$ if we can show that the map

$$j_{r,k}\colon \mathcal{E}^{r+k}_c(U)^\times \to C^r_c(U,\mathbb{K}^d), \quad j_{r,k}(\gamma) := \gamma^* := (\mathrm{id}_U+\gamma)^{-1} - \mathrm{id}_U$$

is $C^k_{\mathbb{K}}$ on some open 0-neighbourhood in $\mathcal{E}^{r+k}_c(U)$. Note that $S := \bigoplus_{i\in I}\mathcal{E}^{r+k}_c(U_i)^\times$ is an open subset of $\mathcal{E}^{r+k}_c(U)^\times \subseteq C^{r+k}_c(U,\mathbb{K}^d) = \bigoplus_{i\in I} C^{r+k}(U_i,\mathbb{K}^d)$, and $j_{r,k}|_S = \oplus_{i\in I}\, j_{i,r,k}\colon S \to \bigoplus_{i\in I} C^r(U_i,\mathbb{K}^d) = C^r_c(U,\mathbb{K}^d)$, where $j_{i,r,k} : \mathcal{E}^{r+k}_c(U_i)^\times \to C^r_c(U_i,\mathbb{K}^d)$, $j_{i,r,k}(\gamma) := (\mathrm{id}_{U_i}+\gamma)^{-1} - \mathrm{id}_{U_i}$. By Proposition 6.9, $\oplus_{i\in I}\, j_{i,r,k}$ will be of class $C^k_{\mathbb{K}}$ if each $j_{i,r,k}$ is of class $C^k_{\mathbb{K}}$. Summing up, in order that $\iota_{r,k}$ be of class $C^k_{\mathbb{K}}$, for all r and k, we only need to establish the following claim:

Claim. For each $i \in I$, $r \in \mathbb{N}\cup\{\infty\}$ and $k\in\mathbb{N}_0\cup\{\infty\}$, the map $j_{i,r,k}$ is of class $C^k_{\mathbb{K}}$ on some open zero-neighbourhood in $\mathcal{E}^{r+k}(U_i)^\times$, where $\mathcal{E}^{r+k}(U_i) := \mathcal{E}^{r+k}_c(U_i)$.

14.15 Fix i, r and k. Since U_i is compact, $\mathrm{Diff}^r_c(U_i) = \mathrm{Diff}^r(U_i)$ is the set of all $C^r_{\mathbb{K}}$-diffeomorphisms of U_i, and the map $C^r(U_i,\mathbb{K}) \to C^r(U_i,\mathbb{K}^d)$, $\gamma\mapsto\gamma+\mathrm{id}_{U_i}$ is an affine-linear homeomorphism and hence a $C^\infty_{\mathbb{K}}$-diffeomorphism, which takes $\mathcal{E}^r(U_i)^\times$ diffeomorphically onto the open subset $\mathrm{Diff}^r(U_i)\subseteq C^r(U_i,\mathbb{K}^d)$. Thus $\mathrm{Diff}^r(U_i)$ simply is an open $C^\infty_{\mathbb{K}}$-submanifold of $C^r(U_i,\mathbb{K}^d)$. Likewise for $\mathrm{Diff}^{r+k}(U_i)$. In order that $j_{i,r,k}$ be $C^k_{\mathbb{K}}$ on some open zero-neighbourhood, it therefore suffices to show that inversion $h\colon P\to C^r(U_i,\mathbb{K}^d)$, $h(\gamma) := \gamma^{-1}$ is $C^k_{\mathbb{K}}$ on the open identity neighbourhood

$$P := \{\mathrm{id}_{U_i}+\sigma\colon \sigma\in\Omega_i\cap C^{r+k}(U_i,\mathbb{K}^d)\}$$

of $\mathrm{Diff}^{r+k}(U_i)$, where $\Omega_i \subseteq C^1(U_i,\mathbb{K}^d)$ is as in **14.11**.[18] We only need to prove that $h^\wedge\colon P\times U_i\to\mathbb{K}^d$, $h^\wedge(\gamma,x) := h(\gamma)(x) = \gamma^{-1}(x)$ is $C^{r+k}_{\mathbb{K}}$; then $h = (h^\wedge)^\vee\colon P\to C^r(U_i,\mathbb{K}^d)$ will be $C^k_{\mathbb{K}}$, by Lemma 12.1 (a). By Lemma 11.1, the evaluation map

$$\varepsilon\colon P\times U_i\to\mathbb{K}^d\,,\quad \varepsilon(\gamma,x) := \gamma(x)$$

is $C^{r+k}_{\mathbb{K}}$. Since $\varepsilon(\gamma,\bullet) = \gamma$ is a diffeomorphism of U_i for each $\gamma\in P$, and $C^{r+k}(U_i,\mathbb{K}^d)$ is metrizable (Proposition 4.19 (c)), the Inverse Function Theorem with Parameters [28,

[18]The discussion in **14.11** shows that indeed $P\subseteq\mathrm{Diff}^{r+k}(U_i)$.

Thm. 8.1 (c)′] can be applied to ε, with the diffeomorphism $\gamma \in P$ as the parameter. The theorem shows that $P \times U_i \to \mathbb{K}^d$, $(\gamma, x) \mapsto (\varepsilon(\gamma, \bullet))^{-1}(x) = \gamma^{-1}(x) = h^\wedge(\gamma, x)$ is of class $C^{r+k}_{\mathbb{K}}$. The claim is established.

14.16 Arguing as in **14.8**, we deduce from the fact that $\iota_{r,k}$ is of class $C^k_{\mathbb{K}}$ for all $r, k \in \mathbb{N}$ that $\widetilde{\iota}$ is smooth. This completes the proof of Proposition 14.6. $\square$

Lemma 14.17 *Let $n, m \in \mathbb{N}$, $r \in \mathbb{N}_0 \cup \{\infty\}$, $U \subseteq \mathbb{K}^n$, $V \subseteq \mathbb{K}^m$ be open subsets, $\phi\colon U \to V$ be a proper mapping of class $C^r_{\mathbb{K}}$, and E be a topological $\mathbb{K}$-vector space. Then*

$$C^r_c(\phi, E)\colon C^r_c(V, E) \to C^r_c(U, E), \quad \gamma \mapsto \gamma \circ \phi$$

is a continuous $\mathbb{K}$-linear map.

Proof. Indeed, the mapping $C^r_c(\phi, E)$ being linear, by Proposition 8.13 (c) we only need to show that its restriction to $C^r_K(V, E)$ is continuous, for every compact open subset K of V. Since ϕ is assumed to be proper, $L := \phi^{-1}(K) \subseteq U$ is a compact, open subset. It is clear that $C^r_c(\phi, E)$ takes $C^r_K(V, E)$ into $C^r_L(U, E)$. Using the obvious identifications $C^r_K(V, E) \cong C^r(K, E)$ and $C^r_L(U, E) \cong C^r(L, E)$, the map $C^r_c(\phi, E)|^{C^r_L(U,E)}_{C^r_K(V,E)}$ corresponds to the pullback $C^r(\phi|^K_L, E)\colon C^r(K, E) \to C^r(L, E)$, which is a continuous linear map by Lemma 4.4. $\square$

Lemma 14.18 *Let $U \subseteq \mathbb{K}^d$, $V \subseteq \mathbb{K}^d$ be open subsets and $\phi\colon U \to V$ be a bijection. Then the following holds:*

(a) *Let $r, k \in \mathbb{N}_0 \cup \{\infty\}$, and suppose that ϕ is a $C^{r+k}_{\mathbb{K}}$-diffeomorphism. Then*

$$\Phi\colon \mathrm{End}^r_c(U) \to \mathrm{End}^r_c(V), \quad \Phi(\gamma) := \phi \circ \gamma \circ \phi^{-1}$$

is a $C^k_{\mathbb{K}}$-diffeomorphism and an isomorphism of monoids.

(b) *If ϕ is a $C^\infty_{\mathbb{K}}$-diffeomorphism, then $\mathrm{End}^\infty_c(U)\breve{} \to \mathrm{End}^\infty_c(V)\breve{}$, $\gamma \mapsto \phi \circ \gamma \circ \phi^{-1}$ is a $C^\infty_{\mathbb{K}}$-diffeomorphism and an isomorphism of monoids.*

Proof. (a) It is obvious that Φ is a bijection, whose inverse $\gamma \mapsto \phi^{-1} \circ \gamma \circ \phi$ also is a map as described in the lemma. Furthermore, clearly Φ is a homomorphism of monoids. In view of the preceding it only remains to show that Φ is a $C^k_{\mathbb{K}}$-map, or equivalently, that

$$\Psi_{r,k}\colon \mathcal{E}^r_c(U) \to \mathcal{E}^r_c(V), \quad \Psi_{r,k}(\gamma) := \phi \circ (\mathrm{id}_U + \gamma) \circ \phi^{-1} - \mathrm{id}_V = \phi \circ (\phi^{-1} + \gamma \circ \phi^{-1}) - \mathrm{id}_V$$

is of class $C^k_{\mathbb{K}}$. To see this, we note first that $\Psi_{r,k}$ is almost local. Indeed: Choose any locally finite cover $(U_\ell)_{\ell \in L}$ of U by relatively compact, open sets U_ℓ. Then $V_\ell := \phi(U_\ell)$ defines a locally finite cover $(V_\ell)_{\ell \in L}$ of V by relatively compact, open subsets $V_\ell \subseteq V$. Given any $\ell \in L$, for every $x \in V_\ell$ and $\gamma \in \mathcal{E}^r_c(U)$ we have

$$\Psi_{r,k}(\gamma)(x) = \phi\big(\phi^{-1}(x) + \gamma(\phi^{-1}(x))\big) \; - \; x\,,$$

which only depends on the value of γ at $\phi^{-1}(x) \in U_\ell$. Thus $\Psi_{r,k}$ is almost local.

By the Smoothness Theorem (Theorem 10.4), $\Psi_{r,k}$ will be $C^k_{\mathbb{K}}$ if we can show that the restriction f of $\Psi_{r,k}$ to the open subset $\mathcal{E}^r_c(U) \cap C^r_K(U, \mathbb{K}^d)$ of $C^r_K(U, \mathbb{K}^d)$ is $C^k_{\mathbb{K}}$, for every compact subset $K \subseteq U$. It suffices to show this for K open and compact, which we assume now. The image of f is contained in $C^r_{\phi(K)}(V, \mathbb{K}^d)$. The inclusion mapping $C^r_{\phi(K)}(V, \mathbb{K}^d) \to C^r_c(V, \mathbb{K}^d)$ being continuous linear and hence smooth, it therefore suffices to prove that f is $C^k_{\mathbb{K}}$ as a map into $C^r_{\phi(K)}(V, \mathbb{K}^d)$. Since $\phi(K)$ is compact and open, the restriction map

$$C^r_{\phi(K)}(V, \mathbb{K}^d) \to C^r_{\phi(K)}(\phi(K), \mathbb{K}^d) \;=\; C^r(\phi(K), \mathbb{K}^d)$$

is an isomorphism of topological vector spaces (Lemma 4.24). In order that f be $C^k_{\mathbb{K}}$, we therefore only need to show that

$$g\colon \mathcal{E}^r_c(U) \cap C^r_K(U, \mathbb{K}^d) \to C^r(\phi(K), \mathbb{K}^d), \quad g(\gamma) := f(\gamma)|_{\phi(K)}$$

is $C^k_{\mathbb{K}}$. Note that

$$\begin{aligned} g(\gamma) &= \phi \circ (\mathrm{id}_K + \gamma|_K) \circ \phi^{-1}|_{\phi(K)} \;-\; \mathrm{id}_{\phi(K)} \\ &= \big(C^r(\phi^{-1}|_{\phi(K)}, \mathbb{K}^d) \circ C^r(K, \phi)\big)\,(\underbrace{\mathrm{id}_K + \gamma|_K}_{\in C^r(K,U)}) \;-\; \mathrm{id}_{\phi(K)}\,. \end{aligned} \tag{44}$$

The pullback $C^r(\phi^{-1}|_{\phi(K)}, \mathbb{K}^d)\colon C^r(K, \mathbb{K}^d) \to C^r(\phi(K), \mathbb{K}^d)$ is continuous linear and hence smooth, by Lemma 4.11; the map $C^r_c(U, \mathbb{K}^d) \to C^r(U, \mathbb{K}^d) \to C^r(K, \mathbb{K}^d)$, $\gamma \mapsto \gamma|_K$ composed of inclusion and restriction is continuous linear and hence smooth (cf. Proposition 8.13 (a) and Lemma 4.11); and $C^r(K, \phi)\colon C^r(K, U) \to C^r(K, V)$ is $C^k_{\mathbb{K}}$ as ϕ is $C^{r+k}_{\mathbb{K}}$, by Corollary 4.21. Hence (44) shows that g is $C^k_{\mathbb{K}}$, as required.

(b) Apparently (b) will hold if we can show that the mapping $\mathcal{E}^\infty_c(U)\breve{} \to \mathcal{E}^\infty_c(V)\breve{}$, $\gamma \mapsto \phi \circ (\mathrm{id}_U + \gamma) \circ \phi^{-1} - \mathrm{id}_V$ is $C^k_{\mathbb{K}}$ for all $k \in \mathbb{N}$. But this readily follows from the fact that $\Psi_{k,k}$ is of class $C^k_{\mathbb{K}}$ for all $k \in \mathbb{N}$ (cf. **14.8**). □

Proposition 14.19 *Let $d \in \mathbb{N}$ and $U \subseteq \mathbb{K}^d$ be a non-empty open subset. Then the following holds:*

(a) *For each $r \in \mathbb{N} \cup \{\infty\}$, there is a uniquely determined $C^\infty_{\mathbb{K}}$-manifold structure on the group $\mathrm{Diff}^r(U)$ of all $C^r_{\mathbb{K}}$-diffeomorphisms of U such that $\mathrm{Diff}^r(U)$ becomes a topological group, the right translation maps $R_\gamma : \mathrm{Diff}^r(U) \to \mathrm{Diff}^r(U)$, $R_\gamma(\eta) := \eta \circ \gamma$ are $C^\infty_{\mathbb{K}}$ for each $\gamma \in \mathrm{Diff}^r(U)$, and such that $\mathrm{Diff}^r_c(U)$ is an open $C^\infty_{\mathbb{K}}$-submanifold of $\mathrm{Diff}^r(U)$.*

(b) *For any $r \in \mathbb{N} \cup \{\infty\}$ and $k \in \mathbb{N}_0 \cup \{\infty\}$, the composition map*

$$\mathrm{Diff}^{r+k}(U) \times \mathrm{Diff}^r(U) \to \mathrm{Diff}^r(U), \quad (\gamma, \eta) \mapsto \gamma \circ \eta \tag{45}$$

and the inversion map

$$\mathrm{Diff}^{r+k}(U) \to \mathrm{Diff}^r(U), \quad \gamma \mapsto \gamma^{-1} \tag{46}$$

are of class $C^k_{\mathbb{K}}$*, with respect to the smooth manifold structures from* (a)*.*

(c) *There is a uniquely determined smooth manifold structure on* $\mathrm{Diff}^\infty(U)$ *turning it into a* $\mathbb{K}$*-Lie group and such that* $\mathrm{Diff}^\infty_c(U)$ *is an open smooth submanifold of* $\mathrm{Diff}^\infty(U)$*.*

(d) *There is a uniquely determined smooth manifold structure on* $\mathrm{Diff}^\infty(U)$ *turning it into a* $\mathbb{K}$*-Lie group* (*which we denote by* $\mathrm{Diff}^\infty(U)\tilde{}$)*, such that* $\mathrm{Diff}^\infty_c(U)\tilde{}$ *is an open smooth submanifold of* $\mathrm{Diff}^\infty(U)\tilde{}$*.*

Proof. It readily follows from Proposition 14.6 (d) that $\mathrm{Diff}^r_c(U)$ is a normal subgroup of $\mathrm{Diff}^r(U)$, for each $r \in \mathbb{N} \cup \{\infty\}$. Given $\gamma \in \mathrm{Diff}^r(U)$, let $I_\gamma \colon \mathrm{Diff}^r_c(U) \to \mathrm{Diff}^r_c(U)$ denote the automorphism of groups $\eta \mapsto \gamma \circ \eta \circ \gamma^{-1}$.

(a) Given $\gamma \in \mathrm{Diff}^r(U)$, consider the map

$$\kappa_\gamma \colon \mathrm{Diff}^r_c(U) \circ \gamma \to \mathcal{E}^r_c(U)^\times\,, \quad \kappa_\gamma(\eta) := \beta_r^{-1}(\eta \circ \gamma^{-1})\,.$$

We claim that $\mathcal{A} := \{\kappa_\gamma : \gamma \in \mathrm{Diff}^r(U)\}$ is an atlas defining a $C^\infty_{\mathbb{K}}$-manifold structure on $\mathrm{Diff}^r(U)$ (equipped with the final topology with respect to the family of the maps $\kappa_\gamma^{-1} : \mathcal{E}^r_c(U)^\times \to \mathrm{Diff}^r(U)$). The domains of the maps κ_γ cover $\mathrm{Diff}^r(U)$. Let us prove compatibility of the charts. If $\gamma, \overline{\gamma} \in \mathrm{Diff}^r(M)$ such that $\mathrm{Diff}^r_c(U) \circ \gamma$ and $\mathrm{Diff}^r_c(U) \circ \overline{\gamma}$ have non-empty intersection, then $\overline{\gamma} \circ \gamma^{-1} \in \mathrm{Diff}^r_c(U)$ and the two cosets coincide. For $\eta \in \mathcal{E}^r_c(U)^\times$, we have

$$(\kappa_\gamma \circ \kappa_{\overline{\gamma}}^{-1})(\eta) = \beta_r^{-1}(\beta_r(\eta) \circ \overline{\gamma} \circ \gamma^{-1}) = \beta_r^{-1}\big(\rho_{r,\overline{\gamma}\circ\gamma^{-1}}(\beta_r(\eta))\big)$$

using right translation $\rho_{r,\overline{\gamma}\circ\gamma^{-1}}$ on $\mathrm{Diff}^r_c(U)$, which is smooth. Hence $\kappa_\gamma \circ \kappa_{\overline{\gamma}}^{-1}$ is smooth, as required for compatibility. Now standard arguments provide a smooth manifold structure on $\mathrm{Diff}^r(U)$ with atlas $\mathcal{A}$. Since $\kappa_{\mathrm{id}} = \beta_r^{-1} : \mathrm{Diff}^r_c(U) \to \mathcal{E}^r_c(U)^\times$, we see that $\mathrm{Diff}^r_c(U)$ is an open submanifold of $\mathrm{Diff}^r(U)$. Given $\gamma \in \mathrm{Diff}^r(U)$, for each $\eta \in \mathrm{Diff}^r(U)$ we have $\big(\kappa_{\eta\circ\gamma}\big)^{-1} \circ R_\gamma \circ \kappa_\eta^{-1} = \mathrm{id}$ on $\mathcal{E}^r_c(U)^\times$, entailing that R_γ is smooth. The topology underlying $\mathrm{Diff}^r(U)$ makes it a topological group, because it has the following properties (cf. [39, Thm. 4.5]): the topological group $\mathrm{Diff}^r_c(U)$ is an open subgroup of $\mathrm{Diff}^r(U)$; all right translations are homeomorphisms of $\mathrm{Diff}^r(U)$; and I_γ is continuous for each $\gamma \in \mathrm{Diff}^r(U)$, by Lemma 14.18.

(b) Let $\gamma \in \mathrm{Diff}^{r+k}(U)$, $\eta \in \mathrm{Diff}^r(U)$. For all $\overline{\gamma} \in \mathrm{Diff}^{r+k}_c(U)$ and $\overline{\eta} \in \mathrm{Diff}^r_c(U)$, we have

$$(\overline{\gamma} \circ \gamma) \circ (\overline{\eta} \circ \eta) = \overline{\gamma} \circ (\gamma \circ \overline{\eta} \circ \gamma^{-1}) \circ (\gamma \circ \eta)\,.$$

Right translation by $\gamma \circ \eta$ being smooth, $I_\gamma \colon \mathrm{Diff}^r_c(U) \to \mathrm{Diff}^r_c(U)$ being $C^k_{\mathbb{K}}$ (Lemma 14.18) and composition $\mathrm{Diff}^{r+k}_c(U) \times \mathrm{Diff}^r_c(U) \to \mathrm{Diff}^r_c(U)$ being $C^k_{\mathbb{K}}$ (Proposition 14.6), the preceding formula defines a $C^k_{\mathbb{K}}$-function $\mathrm{Diff}^{r+k}_c(U) \times \mathrm{Diff}^r_c(U) \to \mathrm{Diff}^r(U)$ of $(\bar{\gamma}, \bar{\eta})$. Hence the

composition map (45) is $C^k_{\mathbb{K}}$ on the open neighbourhood $(\mathrm{Diff}^{r+k}_c(U)\circ\gamma)\times(\mathrm{Diff}^r_c(U)\circ\eta)$ of (γ,η). Similarly, the inversion map (46) is $C^k_{\mathbb{K}}$ on the open neighbourhood $\mathrm{Diff}^{r+k}_c(U)\circ\gamma$ of γ because

$$(\overline{\gamma}\circ\gamma)^{-1}=\gamma^{-1}\circ\overline{\gamma}^{-1}=\gamma^{-1}\circ\overline{\gamma}^{-1}\circ\gamma\circ\gamma^{-1}=(I_{\gamma^{-1}}(\overline{\gamma}^{-1}))\circ\gamma^{-1},$$

where inversion $\mathrm{Diff}^{r+k}_c(U)\to\mathrm{Diff}^r_c(U)$ is $C^k_{\mathbb{K}}$ by Proposition 14.6 (e) and $I_{\gamma^{-1}}$ is $C^k_{\mathbb{K}}$ by Lemma 14.18 (a).

(c) By (b), the $C^\infty_{\mathbb{K}}$-manifold structure from (a) makes $\mathrm{Diff}^\infty(U)$ a Lie group.

(d) By Lemma 14.18 (b), the automorphism I_γ of $\mathrm{Diff}^r_c(U)\breve{}$ is $C^\infty_{\mathbb{K}}$, for any $\gamma\in\mathrm{Diff}^r(U)$. Therefore Part (d) directly follows from Proposition 1.18. □

Definition 14.20 Let $\mathbb{K}$ be a local field, $r\in\mathbb{N}\cup\{\infty\}$, and M be a σ-compact $\mathbb{K}$-manifold of class $C^\infty_{\mathbb{K}}$, of finite, positive dimension d. By Lemma 8.3 (a), there exists a $C^\infty_{\mathbb{K}}$-diffeomorphism $\psi\colon M\to U_\psi$ from M onto an open subset $U_\psi\subseteq\mathbb{K}^d$. Then

$$\Theta_\psi\colon\mathrm{Diff}^r(M)\to\mathrm{Diff}^r(U_\psi),\quad\Theta_\psi(\xi):=\psi\circ\xi\circ\psi^{-1}$$

is an isomorphism of groups. The map

$$\Xi\colon C^r_c(U_\psi,\mathbb{K}^d)\to C^r_c(M,TM),\quad\Xi(f)(x):=T\psi^{-1}\big(\psi(x),f(\psi(x))\big)$$

being an isomorphism of topological vector spaces, there exists a uniquely determined $C^\infty_{\mathbb{K}}$-manifold structure on $\mathrm{Diff}^r(M)$, modeled on $C^r_c(M,TM)$, which makes the bijection Θ_ψ a $C^\infty_{\mathbb{K}}$-diffeomorphism. The charts of $\mathrm{Diff}^r(M)$ are of the form

$$\kappa_\psi\colon\mathrm{Diff}^r(M)\to C^r_c(M,TM),\quad\kappa_\psi(\gamma):=\Xi(\kappa(\Theta_\psi(\gamma))),$$

for $\kappa\colon P_\kappa\to Q_\kappa$ ranging through the charts of $\mathrm{Diff}^r(U_\psi)$. If $r=\infty$ here, then apparently $\mathrm{Diff}^\infty(M)$ is a Lie group, and Θ_ψ is an isomorphism of Lie groups. Analogously, we make $\mathrm{Diff}^\infty(M)\breve{}$ a Lie group modeled on $C^\infty_c(M,TM)\breve{}:=\varprojlim_{k\in\mathbb{N}_0}C^k_c(M,TM)$.

Proposition 14.21 *Let $\mathbb{K}$ be a local field and M be a σ-compact $C^\infty_{\mathbb{K}}$-manifold of finite, positive dimension d.*

(a) *For each $r\in\mathbb{N}\cup\{\infty\}$, the $C^\infty_{\mathbb{K}}$-manifold structure on $\mathrm{Diff}^r(M)$ is independent of the choice of ψ in Definition 14.20. It makes $\mathrm{Diff}^r(M)$ a topological group and the right translation maps $\mathrm{Diff}^r(M)\to\mathrm{Diff}^r(M)$, $\eta\mapsto\eta\circ\gamma$ are smooth for each $\gamma\in\mathrm{Diff}^r(M)$. Furthermore, for any $k\in\mathbb{N}_0\cup\{\infty\}$, both the composition map*

$$\mathrm{Diff}^{r+k}(M)\times\mathrm{Diff}^r(M)\to\mathrm{Diff}^r(M)$$

and the inversion map $\mathrm{Diff}^{r+k}(M)\to\mathrm{Diff}^r(M)$ are $C^k_{\mathbb{K}}$.

(b) *The $C^\infty_{\mathbb{K}}$-manifold structure on* $\mathrm{Diff}^\infty(M)$ *(resp.,* $\mathrm{Diff}^\infty(M)^\sim$*) is independent of the choice of ψ in Definition* 14.20*; it makes* $\mathrm{Diff}^\infty(M)$ *(resp.,* $\mathrm{Diff}^\infty(M)^\sim$*) a $\mathbb{K}$-Lie group.*

Proof. We only need to show that the manifold structures are independent of the choice of ψ; all other assertions are immediate consequences of Proposition 14.19.

(a) If both $\phi\colon M \to U_\phi$ and $\psi\colon M \to U_\psi$ are $C^\infty_{\mathbb{K}}$-diffeomorphisms onto open subsets of $\mathbb{K}^d$, then $\Theta_\phi \circ (\Theta_\psi)^{-1} : \mathrm{Diff}^r(U_\psi) \to \mathrm{Diff}^r(U_\phi)$, $\xi \mapsto (\phi \circ \psi^{-1}) \circ \xi \circ (\phi \circ \psi^{-1})^{-1}$ is an isomorphism of groups which takes $\mathrm{Diff}^r_c(U_\psi)$ $C^\infty_{\mathbb{K}}$-diffeomorphically onto $\mathrm{Diff}^r_c(U_\phi)$, by Lemma 14.18 (a). Since right translations in the groups $\mathrm{Diff}^r(U_\phi)$ and $\mathrm{Diff}^r(U_\psi)$ are smooth and the homomorphism $f := \Theta_\phi \circ (\Theta_\psi)^{-1}$ is smooth on an open identity neighbourhood, the usual argument shows that the homomorphism f is smooth. Interchanging the roles of ϕ and ψ, we see that also f^{-1} is smooth.

(b) In view of Lemma 14.18 (b), the same argument applies to $\mathrm{Diff}^\infty(M)^\sim$. □

Remark 14.22 If M is a σ-compact, finite-dimensional $C^r_{\mathbb{K}}$-manifold for some $r \in \mathbb{N}$ (but not smooth), we can still use the same arguments to make $\mathrm{Diff}^r(M)$ a topological group and $C^0_{\mathbb{K}}$-manifold modeled on $C^0_c(M, TM)$.[19]

A Proof of Proposition 4.19

In this section, we prove the properties of function spaces asserted in Proposition 4.19. In view of Remark 4.2 (a) and Lemma 4.12 (applied with a countable cover of coordinate neighbourhoods in case of (c)), it suffices to prove assertions (a), (b) and (c) of Proposition 4.19 when $r \in \mathbb{N}_0$ and $M = U$ is an open subset of Z, which we assume now.[20]

(a) The proof is by induction on $r \in \mathbb{N}_0$. Let us assume first that E is complete. If $r = 0$, suppose that (γ_α) is a Cauchy net in $C(U,E)_{c.o.}$. Then $(\gamma_\alpha(x))$ is a Cauchy net in E for each fixed element $x \in U$ and hence convergent, to $\gamma(x) \in E$, say. For each compact subset $K \subseteq U$, the restrictions $\gamma_\alpha|_K$ converge uniformly to $\gamma|_K$, whence $\gamma|_K$ is continuous. Hence γ is continuous, using that U (being open in the k-space Z) is a k-space. Furthermore, $\gamma_\alpha \to \gamma$ in $C(U,E)_{c.o.}$.

Induction step: Assume the assertion is correct for some r. Then both $C(U,E)$ and $C^r(U^{[1]},E)$ are complete and hence so is the topological vector space $C^{r+1}(U,E)$, being isomorphic to a closed vector subspace of $C(U,E) \times C^r(U^{[1]},E)$ by Lemma 4.3. This completes the induction.

If E is merely sequentially complete (resp., Mackey complete) the sequential completeness

[19] Although we obtain an isomorphism of topological groups onto the $C^\infty_{\mathbb{K}}$-manifold $\mathrm{Diff}^r(U_\psi)$ for each $C^r_{\mathbb{K}}$-diffeomorphism $\psi\colon M \to U_\psi \subseteq \mathbb{K}^d$, we cannot expect anymore that the corresponding $C^\infty_{\mathbb{K}}$-manifold structures on $\mathrm{Diff}^r(M)$ are independent of the choice of ψ.

[20] For (a), note that cartesian products and closed vector subspaces of Mackey complete topological vector spaces are Mackey complete.

of $C^r(U,E)$ can be proved in the same way, replacing Cauchy nets by Cauchy sequences (resp., Mackey-Cauchy sequences).[21]

(b) Assume that $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ (resp., that $\mathbb{K}$ is an ultrametric field, with valuation ring $\mathbb{O}$) and E is locally convex. If $K \subseteq U$ is compact and $V \subseteq E$ is a convex, open 0-neighbourhood (resp., an open $\mathbb{O}$-submodule submodule), then apparently also the open 0-neighbourhood $\lfloor K, V \rfloor \subseteq C(U,E)$ is convex (resp., an $\mathbb{O}$-submodule). Hence $C(U,E)$ is locally convex. Likewise, $C(U^{[j]},E)$ is locally convex for each j and hence so is $C^r(U,E)$, its topology being initial with respect to linear maps into the spaces $C(U^{[j]},E)$ for $j \leq r$ (by definition).

(c) Case $r = 0$: There exists an ascending sequence $(K_j)_{j\in\mathbb{N}}$ of compact subsets K_j of U such that K_j is contained in the interior of K_{j+1} for each $j \in \mathbb{N}$, and $U = \bigcup_{j\in\mathbb{N}} K_j$. Then every compact subset of U is contained in K_j for some j, entailing that the topology of uniform convergence on compact sets on $C(U,E)$ is the topology making the map $C(U,E) \to \prod_{j\in\mathbb{N}} C(K_j,E)$, $\gamma \mapsto (\gamma|_{K_j})_{j\in\mathbb{N}}$ a topological embedding, where $C(K_j,E)$ is equipped with the topology of uniform convergence. If $(V_n)_{n\in\mathbb{N}}$ is a countable basis of open 0-neighbourhoods in E, then $(\lfloor K_j, V_n \rfloor)_{n\in\mathbb{N}}$ is a countable basis of open 0-neighbourhoods for $C(K_j,E)$, entailing that this space is metrizable. We readily deduce that also $C(U,E)$ is metrizable.

Suppose that $r \in \mathbb{N}$ now, and suppose that the assertion of the lemma is correct for $r-1$. By Lemma 4.3, $C^r(U,E)$ is isomorphic to a topological vector subspace of $C(U,E) \times C^{r-1}(U^{[1]},E)$. The factors of the product being metrizable by induction, also the product is metrizable and hence so is $C^r(U,E)$. This completes the proof of (c).

(d) Given r, M and E as described in the proposition, let us write $C^r(M,E)_D$ for $C^r(M,E)$, equipped with the initial topology with respect to the family $(D^j)_{\mathbb{N}\ni j\geq r}$ of the mappings $D^j : C^r(M,E) \to C(T^jM,E)_{c.o.}$, $\gamma \mapsto D^j\gamma$, defined as follows: we set $D^0\gamma := \gamma$, let $D^1\gamma := D\gamma := d\gamma \colon TM \to E$ be the second component of the tangent map $T\gamma : TM \to TE = E \times E$, $T_xM \ni v \mapsto (\gamma(x), d\gamma(v))$, and define $D^j\gamma := D(D^{j-1}\gamma) : T^jM := T(T^{j-1}M) \to E$ recursively. Before we establish Proposition 4.19 (d), let us first recall some useful properties of the spaces $C^r(M,E)_D$:

Lemma A.1 *In the situation of Proposition* 4.19 (d)*, we have:*

(a) *If $r = \infty$, then $C^\infty(M,E)_D = \varprojlim_{k\in\mathbb{N}_0} C^k(M,E)_D$ as a topological vector space, with the inclusion maps $C^\infty(M,E)_D \to C^k(M,E)_D$ as the limit maps.*

(b) *If $(U_i)_{i\in I}$ is an open cover of M, then the topology on $C^r(M,E)_D$ is initial with respect to the family $(\rho_i)_{i\in I}$ of restriction maps $\rho_i \colon C^r(M,E) \to C^r(U_i,E)_D$, $\rho_i(\gamma) := \gamma|_{U_i}$.*

(c) *If $\phi \colon M \to U \subseteq Z$ is a $C^r_{\mathbb{R}}$-diffeomorphism, then $C^r(\phi,E) \colon C^r(U,E)_D \to C^r(M,E)_D$, $\gamma \mapsto \gamma \circ \phi$ is an isomorphism of topological $\mathbb{K}$-vector spaces.*

[21] Note that continuous linear maps take Mackey-Cauchy sequences to Mackey-Cauchy sequences because they take bounded sets to bounded sets.

(d) *If $r \in \mathbb{N}_0$, then the topology on $C^{r+1}(M,E)_D$ is initial with respect to the maps $\beta_1 : C^{r+1}(M,E) \to C(M,E)_{c.o.}$, $\beta_1(\gamma) := \gamma$ and $\beta_2 : C^{r+1}(M,E) \to C^r(TM,E)_D$, $\beta_2(\gamma) := D\gamma$.*

Proof. (a) and (d) are immediate from the definition of the topologies. For a proof of (b), see [27, Prop. 24.10] (for example). For (c), see [27, Prop. 24.8].[22] See also [32]. □

In view of Lemma 4.12, Lemma 4.11 and their analogues compiled in Lemma A.1 (b) and (c), Proposition 4.19 (d) will hold in general if we can prove it in the special case when $U := M \subseteq Z$ is an open subset of the modeling space, which we assume now. In view of Remark 4.2 (a) and Lemma A.1 (a), we may also assume that $r \in \mathbb{N}_0$. The following four lemmas will enable us to complete the proof:

Lemma A.2 *For Z, E, $U \subseteq Z$ as before and each $r \in \mathbb{N}_0$, the topology on $C^r(U,E)_D$ is initial with respect to the family $(d^j)_{r \geq j \in \mathbb{N}_0}$, where $d^j : C^r(U,E) \to C(U \times Z^j, E)_{c.o.}$, $\gamma \mapsto d^j\gamma$ is as in* **1.9**.

Proof. Note first that $d^j\gamma$ is a partial map of $D^j\gamma$: There is an injective map $\kappa : \{1,\ldots,j\} \to \{1,\ldots,2^j-1\}$ (independent of Z, U and E), such that

$$d^j\gamma(x,y) \;=\; D^j\gamma(x,\phi(y)) \quad \text{for all } \gamma \in C^r(U,E),\ x \in U \text{ and } y \in Z^j,$$

where $\phi\colon Z^j \to Z^{2^j-1}$ is the (continuous linear) map with kth component

$$\mathrm{pr}_k(\phi(y_1,\ldots,y_j)) \;=\; \begin{cases} y_i & \text{if } \kappa(i) = k \\ 0 & \text{else,} \end{cases}$$

for $k = 1,\ldots,2^j-1$ and $y_1,\ldots,y_j \in Z$ [18, Claim 2, p. 50]. Accordingly, the map $d^j = C(\mathrm{id}_U \times \phi, E) \circ D^j \colon C^r(U,E)_D \to C(U \times Z^j, E)_{c.o.}$ is a composition of D^j and a pullback along a continuous map, and thus d^j is continuous on $C^r(U,E)_D$, for each $j \leq r$.

We now show that, conversely, each D^j is continuous on $C^r(U,E)_d$, i.e., on $C^r(U,E)$, equipped with the topology initial with respect to the family $(d^j)_{j\leq r}$. To this end, we recall that

$$D^j\gamma(x,y_1,\ldots,y_{2^j-1}) \;=\; \sum_{\ell=1}^{j} \sum_{1\leq i_1<i_2<\cdots<i_\ell\leq 2^j-1} c_{i_1,\ldots,i_\ell}\, d^\ell\gamma(x,y_{i_1},\ldots,y_{i_\ell})$$

for all $\gamma \in C^r(U,E)$, $x \in U$ and $y_1,\ldots,y_{2^j-1} \in Z$, for suitable numbers $c_{i_1,\ldots,i_\ell} \in \mathbb{N}_0$ which are independent of Z, U, E, γ, x, and $y_1,\ldots,y_{2j-1}$ (cf. [18, Eqn. (3)]). Hence D^j is a sum of terms of the form $C(\mathrm{id}_U \times \psi_{i_1,\ldots,i_\ell}, E) \circ d^\ell$ with a suitable continuous (linear) map $\psi_{i_1,\ldots,i_\ell}\colon E^{2^j-1} \to E^\ell$, and hence D^j is continuous on $C^r(U,E)_d$. Thus $C^r(U,E)_D = C^r(U,E)_d$. □

[22] The cited results are formulated in [27] only for real manifolds, but they carry over to complex manifolds, with identical proofs.

Lemma A.3 *Let Z, $U \subseteq Z$ and E be as before, and F be a locally convex topological $\mathbb{K}$-vector space such that E is a topological vector subspace of F. The the inclusion maps*

$$C^r(U,E) \to C^r(U,F) \quad \textit{and} \quad C^r(U,E)_D \to C^r(U,F)_D$$

are topological embeddings.

Proof. In view of the definition of the topologies, the assertion readily follows from the well-known (and apparent) fact that the inclusion map $C(Y,E)_{c.o.} \to C(Y,F)_{c.o.}$ is a topological embedding, for any topological space Y. (Apply this with $Y := U^{[j]}$, resp., $Y := U \times Z^j$, for all $j \in \mathbb{N}_0$ such that $j \leq r$). □

Lemma A.4 *Let Z, $U \subseteq Z$ and E be as before. Assume that E is complete. Define*

$$\lambda : C(U \times [0,1], E) \to C(U,E)$$

via $\lambda(\gamma)(x) := \int_0^1 \gamma(x,t)\, dt$ for $\gamma \in C(U \times [0,1], E)$ and $x \in U$. Then λ is a continuous $\mathbb{K}$-linear map.

Proof. The maps $\lambda(\gamma) : U \to E$ are in fact continuous, being parameter-dependent integrals with continuous integrands (see, for example, [27, La. 6.15] or [35]). As clearly λ is linear, it only remains to show that λ is continuous at 0. To this end, let $V \subseteq C(U,E)$ be a 0-neighbourhood. Then there exists a compact subset $K \subseteq U$ and a closed, convex 0-neighbourhood $W \subseteq E$ such that $\lfloor K, W \rfloor := \{\gamma \in C(U,E) : \gamma(K) \subseteq W\} \subseteq V$. Set $I := [0,1]$. Then $\lfloor K \times I, W \rfloor := \{\gamma \in C(U \times I, E) \colon \gamma(K \times I) \subseteq W\}$ is a 0-neighbourhood such that $\lambda(\lfloor K \times I, W \rfloor) \subseteq \lfloor K, W \rfloor \subseteq V$ (cf. [18, La. 1.7]). Hence λ is continuous. □

Lemma A.5 *Given $r \in \mathbb{N}_0$ and an open neighbourhood I of $[0,1]$ in $\mathbb{F}$, consider the set $\Omega \subseteq C^r(U \times I, E)_D$ of all $\gamma \in C^r(U \times I, E)_D$ such that the weak integrals*

$$\iota(\gamma)(x) \ := \int_0^1 \gamma(x,t)\, dt$$

exist in E for all $x \in U$, as well as the weak integrals $\int_0^1 d_1^j \gamma(x,t,y)\, dt$, for all $j \leq r$ and $(x,y) \in U \times E^j$. Here $d_1^j \gamma(x,t,y) := d^j(\gamma(\bullet,t))(x,y)$ for all $x \in U$, $t \in I$, and $y = (y_1, \ldots, y_j) \in Z^j$, or, more explicitly:

$$d_1^j \gamma(x,t,y_1,\ldots,y_j) \ = \ d^j \gamma((x,t), (y_1,0), \ldots, (y_j,0))\,. \tag{47}$$

Then $\iota(\gamma) \in C^r(U,E)$ for all $\gamma \in \Omega$, and

$$\iota : \ C^r(U \times I, E)_D \supseteq \Omega \ \to \ C^r(U,E)_D \tag{48}$$

is a continuous $\mathbb{K}$-linear map.

Proof. (e) By [27, Prop. 8.7] (or [3, La. 7.5]) and [3, Prop. 7.4],[23] the map $\iota(\gamma)$ is $C^r_{\mathbb{F}}$, with

$$d^j\big(\iota(\gamma)\big)(x,y) \;=\; \int_0^1 d_1^j\gamma(x,t,y)\,dt \quad \text{for all } j\leq r,\ x\in U \text{ and } y\in E^j. \tag{49}$$

Clearly ι is linear. To prove the continuity of ι, after passing to the completion of E we may assume that E is complete, and hence $\Omega = C^r(U\times I,E)$, for convenience (cf. Lemma A.3). The topology on $C^r(U,E)_D$ being initial with respect to the maps $d^j\colon C^r(U,E)\to C(U\times Z^j,E)_{c.o.}$ by Lemma A.2, the mapping ι will be continuous if $d^j\circ\iota\colon C^r(U\times I,E)_D\to C(U\times Z^j,E)$, $\gamma\mapsto d^j\iota(\gamma)$ is continuous for $j\leq r$. By (49) and (47), $d^j\circ\iota$ is a composition of the continuous map $d^j\colon C^r(U\times I,E)_D\to C((U\times I)\times(Z\times\mathbb{F})^j,E)$, the continuous pullback $C(f,E)\,:\,C((U\times I)\times(Z\times\mathbb{F})^j,E)\;\to\;C(U\times I\times Z^j,E)$, $\eta\mapsto\eta\circ f$ with $f(x,t,y_1,\ldots,y_j):=(x,t,\,(y_1,0),\ldots,(y_j,0))$, and the integration map $\lambda_j\colon C(U\times I\times Z^j,E)\to C(U\times Z^j,E)$, $\lambda_j(\eta)(x,y):=\int_0^1\eta(x,t,y)\,dt$, which is continuous as a consequence of Lemma A.4. Hence $d^j\circ\iota$ is continuous for each $j\leq r$, and hence so is ι, as asserted. □

We are now in the position to complete the proof of Proposition 4.19 (d). The proof is by induction on $r\in\mathbb{N}_0$. The case $r=0$ is trivial: By definition, both $C^0(U,E)$ and $C^0(U,E)_D$ are equipped with the compact-open topology, and hence coincide.

Induction step: Assume that Proposition 4.19 (d) holds for some $r\in\mathbb{N}_0$. Consider the map $f\colon C^{r+1}(U,E)\to C^{r+1}(U,E)_D$, $f(\gamma):=\gamma$. For β_1 and β_2 as in Lemma A.1 (d), the composition

$$\beta_1\circ f\colon C^{r+1}(U,E)\to C(U,E)_{c.o.}\,,\quad \gamma\mapsto\gamma$$

is continuous by Remark 4.2 (b), and also

$$\beta_2\circ f\colon C^{r+1}(U,E)\to C^r(TU,E)_D\,,\quad \gamma\mapsto d\gamma=\gamma^{[1]}(\bullet,0)$$

is continuous as it is a composition of the continuous map $(\bullet)^{[1]}\colon C^{r+1}(U,E)\to C^r(U^{[1]},E)$ (Remark 4.2 (b)), the restriction map $C^r(U^{[1]},E)\to C^r(TU,E)$ which is a pullback and hence continuous (Lemma 4.4), and the identity map $C^r(TU,E)\to C^r(TU,E)_D$ which is an isomorphism of topological vector spaces by induction. The topology on $C^{r+1}(U,E)_D$ being initial with respect to the maps β_j ($j\in\{1,2\}$), the continuity of the compositions $\beta_j\circ f$ entails that f is continuous.

It remains to show that also f^{-1} is continuous. In view of Remark 4.2 (b), we only need to show that $\alpha_1\circ f^{-1}$ and $\alpha_2\circ f^{-1}$ are continuous, where $\alpha_1\colon C^{r+1}(U,E)\to C(U,E)$ is the inclusion map and $\alpha_2\colon C^{r+1}(U,E)\to C^r(U^{[1]},E)$, $\alpha_2(\gamma):=\gamma^{[1]}$. Here $\alpha_1\circ f^{-1}=\beta_1$ is continuous. To see that $\alpha_2\circ f^{-1}$ is continuous, note that for any $x\in U$ and $y\in Z$, we find an open balanced 0-neighbourhood $J_{(x,y)}\subseteq\mathbb{F}$ and open neighbourhoods $V_{(x,y)}\subseteq U$ of x and $W_{(x,y)}\subseteq Z$ of y such that

$$V_{(x,y)}+2\,J_{(x,y)}\,W_{(x,y)}\;\subseteq\;U \tag{50}$$

[23]Erratum: In the complex case, I should be an open neighbourhood of $[0,1]$ in $\mathbb{C}$ in [3, La. 7.5]. Similar apparent adaptations are necessary in the proof of [3, Prop. 7.4].

(and thus $V_{(x,y)} \times W_{(x,y)} \times 2J_{(x,y)} \subseteq U^{]1[}$). Note that $U^{[1]}$ is covered by $U^{]1[}$, together with the sets $Y_{(x,y)} := V_{(x,y)} \times W_{(x,y)} \times J_{(x,y)}$ for $(x,y) \in U \times Z$. Hence, by Lemma 4.12, the map $\alpha_2 \circ f^{-1}$ will be continuous if we can show that

$$\phi : C^{r+1}(U,E)_D \to C^r(U^{]1[},E)\,, \quad \phi(\gamma) := \gamma^{[1]}|_{U^{]1[}} = \gamma^{]1[}$$

is continuous, as well as the mappings

$$\psi_{(x,y)} : C^{r+1}(U,E)_D \to C^r(Y_{(x,y)},E)\,, \quad \psi_{(x,y)}(\gamma) := \gamma^{[1]}|_{Y_{(x,y)}}\,,$$

for all $(x,y) \in U \times Z$. The formula $\phi(\gamma)(x,y,t) = \frac{1}{t}(\gamma(x+ty) - \gamma(x))$ shows that ϕ is built up from the following continuous maps and hence continuous: 1. The map $C^{r+1}(U,E)_D \to C^r(U^{]1[},E)$, $\gamma \mapsto [(x,y,t) \mapsto \gamma(x+ty)]$. This map is the composition of the inclusion map $C^{r+1}(U,E)_D \to C^r(U,E)_D = C^r(U,E)$ (which is continuous by definition of the D-topologies and the induction hypothesis) and the pullback $C^r(g,E)\colon C^r(U,E) \to C^r(U^{]1[},E)$ along $g\colon U^{]1[} \to U$, $g(x,y,t) := x+ty$, which is continuous by Lemma 4.4. 2. The map $C^{r+1}(U,E)_D \to C^r(U^{]1[},E)$, $\gamma \mapsto [(x,y,t) \mapsto \gamma(x)]$, which is continuous by the same argument. 3. The addition map of the topological vector space $C^r(U^{]1[},E)$. 4. The multiplication operator $m_\tau\colon C^r(U^{]1[},E) \to C^r(U^{]1[},E)$, $\gamma \mapsto \tau \cdot \gamma$ with $\tau\colon U^{]1[} \to \mathbb{F}$, $\tau(x,y,t) := \frac{1}{t}$, which is continuous by Lemma 4.5. Hence ϕ is continuous.

In view of Lemma A.3, it suffices to prove continuity of the maps $\psi_{(x,y)}$ when E is complete, which we assume now (we can always replace E by its completion). For the proof, fix $(x,y) \in U \times Z$; we abbreviate $V := V_{(x,y)}$, $W := W_{(x,y)}$, $J := J_{(x,y)}$, $Y := Y_{(x,y)} = V \times W \times J$, and $\psi := \psi_{(x,y)}$. Let $B_2(0) := B_2^{\mathbb{F}}(0) \subseteq \mathbb{F}$. Then

$$\psi(\gamma)(u,v,t) = \gamma^{[1]}(u,v,t) = \int_0^1 d\gamma(u+stv,v)\,ds \quad \text{for all } (u,v,t) \in Y$$

(see [3, Prop. 7.4] and its proof). We can therefore interpret ψ as a composition

$$\begin{aligned} C^{r+1}(U,E)_D &\to C^r(U \times Z,E)_D = C^r(U \times Z,E) \to C^r(Y \times B_2(0),E) \\ &= C^r(Y \times B_2(0),E)_D \to C^r(Y,E)_D = C^r(Y,E) \end{aligned}$$

of the continuous map $D\colon C^{r+1}(U,E)_D \to C^r(U \times Z,E)_D$ (Lemma A.1 (d)), the continuous pullback $C^r(h,E) : C^r(U \times Z,E) \to C^r(Y \times B_2(0),E)$ along $h : Y \times B_2(0) \to U \times Z$, $h(u,v,t,s) := (u+stv,v)$ (Lemma 4.4), and the integration map

$$\iota : C^r(Y \times B_2(0),E)_D \to C^r(Y,E)_D$$

taking η to the map $(u,v,t) \mapsto \int_0^1 \eta(u,v,t,s)\,ds$.[24] Here ι is continuous by Lemma A.5. Hence ψ is continuous, and hence so is f^{-1}, which completes the proof of Proposition 4.19.□

[24]Note that all of the required weak integrals exist, by completeness of E.

B Proof of Proposition 4.23

Part (a) follows from Part (b) in an obvious way. We therefore only need to prove (b). Furthermore, as in the proof of Proposition 4.16, we may assume that $r, k \in \mathbb{N}_0$.

Reduction to open subsets of X, $\overline{X}$ and Z

As $\mathbb{F}$ is locally compact and X finite-dimensional, we deduce that X is locally compact (cf. [12, I, §2, No. 3, Thm. 2]), and hence so are M and Y. For each $i := (x, \bar{x}) \in Y \times \overline{M} =: I$, we find a chart $\tau_i : W_i \to N_i$ of N around $\sigma(x)$, onto an open subset $N_i \subseteq Z$. Let $\bar{\kappa}_i : \overline{S}_i \to \overline{M}_i \subseteq \overline{X}$ be a chart of $\overline{M}$ around $\bar{x}$, and $\kappa_i : S_i \to M_i \subseteq X$ be a chart of M around x such that $S_i \subseteq \sigma^{-1}(W_i)$. As Y is locally compact, we find a compact neighbourhood C_i of x in Y such that $C_i \subseteq S_i$. We let $U_i := C_i^0$ be the interior of C_i. Define $K_i := \kappa_i(C_i)$, $Y_i := \kappa_i(U_i) = K_i^0$, and $\eta_i := \kappa_i|_{U_i}^{Y_i}$. Then $Y = \bigcup_{i \in I} U_i$, and $\{\eta_i \times \bar{\kappa}_i : i \in I\}$ is an atlas for $Y \times \overline{M}$. Abbreviate $\Theta_i := \theta_{\eta_i \times \bar{\kappa}_i} : C^r(Y \times \overline{M}, F) \to C^r(Y_i \times \overline{M}_i, F)$, $\theta_i := \theta_{\kappa_i} : C^r(M, E) \to C^r(M_i, E)$, and $\bar{\theta}_i := \theta_{\bar{\kappa}_i} : C^r(\overline{M}, E) \to C^r(\overline{M}_i, E)$ (see (13) in **4.7** for the notations). Then

$$\lambda \colon C^r(Y \times \overline{M}, F) \to \prod_{i \in I} C^r(Y_i \times \overline{M}_i, F), \quad \gamma \mapsto (\Theta_i(\gamma))_{i \in I} = (\gamma \circ (\eta_i^{-1} \times \bar{\kappa}_i^{-1}))_{i \in I}$$

is a topological embedding (Lemma 4.9) whose image is easily seen to be closed (cf. Lemma 4.12). Hence, by Lemma 1.15 and Lemma 1.16, the map ϕ will be of class $C^k_{\mathbb{K}}$ if we can show that $\Theta_i \circ \phi$ is of class $C^k_{\mathbb{K}}$ for each $i \in I$. Using the $C^r_{\mathbb{F}}$-map $\sigma_i := \tau_i \circ \sigma|_{S_i}^{W_i} \circ \eta_i^{-1} \colon Y_i \to N_i$ and the $C^{r+k}_{\mathbb{K}}$-map $\tilde{f}_i := \tilde{f} \circ (\tau_i^{-1} \times \mathrm{id}_U \times \mathrm{id}_{\overline{E}} \times \mathrm{id}_P) \colon N_i \times U \times \overline{E} \times P \to F$, we define $f_i := \tilde{f}_i \circ (\sigma_i \times \mathrm{id}_U \times \mathrm{id}_{\overline{E}} \times \mathrm{id}_P) \colon Y_i \times U \times \overline{E} \times P \to F$. For $\gamma \in \lfloor K, U \rfloor_r$, $\bar{\gamma} \in C^r(\overline{M}, \overline{E})$, $x \in Y_i$, $\bar{x} \in \overline{M}_i$, and $p \in P$, we have

$$\begin{aligned} \Theta_i(\phi(\gamma, \bar{\gamma}, p))(x, \bar{x}) &= f(\eta_i^{-1}(x), \gamma(\eta_i^{-1}(x)), \bar{\gamma}(\bar{\kappa}_i^{-1}(\bar{x})), p) = f_i(x, \theta_i(\gamma)(x), \bar{\theta}_i(\bar{\gamma})(\bar{x}), p) \\ &= \phi_i(\theta_i(\gamma), \bar{\theta}_i(\bar{\gamma}), p)(x, \bar{x}), \end{aligned}$$

where

$$\phi_i \colon \lfloor K_i, U \rfloor_r \times C^r(\overline{M}_i, \overline{E}) \times P \to C^r(Y_i \times \overline{M}_i, F), \quad \phi_i(\eta, \bar{\eta}, p)(x, \bar{x}) := f_i(x, \eta(x), \bar{\eta}(\bar{x}), p)$$

with $\lfloor K_i, U \rfloor_r \subseteq C^r(M_i, E)$. Thus $\Theta_i \circ \phi = \phi_i \circ (\theta_i|_{\lfloor K, U \rfloor_r}^{\lfloor K_i, U \rfloor_r} \times \bar{\theta}_i \times \mathrm{id}_P)$, showing that $\Theta_i \circ \phi$ will be of class $C^k_{\mathbb{K}}$ if ϕ_i is of class $C^k_{\mathbb{K}}$. Thus, each M_i being an open subset of X, $\overline{M}_i$ open in $\overline{X}$, and N_i an open subset of Z, the proposition will hold if we can prove (b) in the case where M, $\overline{M}$ and N are open subsets of X, $\overline{X}$ and Z, respectively, which we shall assume for the rest of the proof. We proceed by induction on k.

The case $k = 0$.

The proof is by induction on r. If $r = 0$, then the topology on $C^0(M, E)$, $C^0(\overline{M}, \overline{E})$ and $C^0(Y \times \overline{M}, F)$ is the topology of uniform convergence on compact sets (see **4.8**). Let $\gamma \in \lfloor K, U \rfloor \subseteq C^r(M, E)$, $\bar{\gamma} \in C^r(\overline{M}, \overline{E})$, $p \in P$, L be a compact subset of $Y \times \overline{M}$, and $V \subseteq F$ be an open zero-neighbourhood. Let $W \subseteq F$ be an open zero-neighbourhood such that $W - W \subseteq V$. For each $i := (x, \bar{x}) \in L$, there are open neighbourhoods $A_i \subseteq Y$ and $\overline{A}_i \subseteq \overline{M}$ of x, resp., $\bar{x}$, and open zero-neighbourhoods $B_i \subseteq E$, $\overline{B}_i \subseteq \overline{E}$ and $C_i \subseteq H$ such that $\gamma(A_i) + B_i \subseteq \gamma(K) + B_i \subseteq U$, $p + C_i \subseteq P$, and

$$f(y, u, \bar{u}, q) - f(x, \gamma(x), \bar{\gamma}(x), p) \in W$$

for all $y \in A_i$, $u \in \gamma(A_i) + B_i$, $\bar{u} \in \bar{\gamma}(\overline{A}_i) + \overline{B}_i$, and $q \in p + C_i$. By compactness, $L \subseteq \bigcup_{i \in I}(A_i \times \overline{A}_i)$ for some finite subset $I \subseteq L$. Then $B := \bigcap_{i \in I} B_i \subseteq E$, $\overline{B} := \bigcap_{i \in I} \overline{B}_i \subseteq \overline{E}$ and $C := \bigcap_{i \in I} C_i \subseteq H$ are open zero-neighbourhoods, and $\overline{K} := \mathrm{pr}_2(L) \subseteq \overline{M}$ is compact, where $\mathrm{pr}_2 \colon Y \times \overline{M} \to \overline{M}$ is the coordinate projection. Let $\xi \in \gamma + \lfloor K, B \rfloor \subseteq C^0(M, E)$, $\bar{\xi} \in \bar{\gamma} + \lfloor \overline{K}, \overline{B} \rfloor \subseteq C^0(\overline{M}, \overline{E})$, and $q \in p + C \subseteq P$. Given $(y, \bar{y}) \in L$, there is $i = (x, \bar{x}) \in I$ such that $(y, \bar{y}) \in A_i \times \overline{A}_i$. Then $\xi \in \lfloor K, U \rfloor$, and

$$\begin{aligned}
&f(y, \xi(y), \bar{\xi}(\bar{y}), q) - f(y, \gamma(y), \bar{\gamma}(\bar{y}), p)\\
&\quad = \; f(y, \xi(y), \bar{\xi}(\bar{y}), q) - f(x, \gamma(x), \bar{\gamma}(\bar{x}), p) \; - \; (f(y, \gamma(y), \bar{\gamma}(\bar{y}), p) - f(x, \gamma(x), \bar{\gamma}(\bar{x}), p))\\
&\quad \in \; W - W \; \subseteq \; V\,.
\end{aligned}$$

We have shown that $\phi(\xi, \bar{\xi}, q) - \phi(\gamma, \bar{\gamma}, p) \in \lfloor L, V \rfloor \subseteq C(Y \times \overline{M}, F)$ for all $(\xi, \bar{\xi}, q)$ in the open neighbourhood $(\gamma + \lfloor K, B \rfloor) \times (\bar{\gamma} + \lfloor \overline{K}, \overline{B} \rfloor) \times (p + C)$ of $(\gamma, \bar{\gamma}, p)$. Thus ϕ is continuous.

Induction step on r. We write ϕ_r for ϕ, to emphasize its dependence on r. Suppose the assertion of the lemma is correct for $k = 0$ and some $r \in \mathbb{N}_0$. Suppose that the hypotheses of the lemma are satisfied by $\tilde{f}$ and σ, with r replaced by $r + 1$. The map ϕ_r being continuous, we see as in the proof of Proposition 4.16 that ϕ_{r+1} will be continuous if we can show that the map

$$\psi \colon \lfloor K, U \rfloor_{r+1} \times C^{r+1}(\overline{M}, \overline{E}) \times P \to C^r((Y \times \overline{M})^{[1]}, F), \quad \psi(\gamma, \bar{\gamma}, p) := \phi_{r+1}(\gamma, \bar{\gamma}, p)^{[1]}$$

is continuous. Here we have, by the Chain Rule,

$$\begin{aligned}
&\psi(\gamma, \bar{\gamma}, p)(x, \bar{x}, y, \bar{y}, t)\\
&\qquad = \; \phi_{r+1}(\gamma, \bar{\gamma}, p)^{[1]}(x, \bar{x}, y, \bar{y}, t)\\
&\qquad = \; \tilde{f}^{[1]}((\sigma(x), \gamma(x), \bar{\gamma}(\bar{x}), p), \; (\sigma^{[1]}(x, y, t), \gamma^{[1]}(x, y, t), \bar{\gamma}^{[1]}(\bar{x}, \bar{y}, t), 0), \; t)
\end{aligned} \tag{51}$$

for $(x, \bar{x}, y, \bar{y}, t) \in (Y \times \overline{M})^{[1]}$. Let $(\gamma_0, \bar{\gamma}_0, p_0) \in \lfloor K, U \rfloor_{r+1} \times C^{r+1}(\overline{M}, \overline{E}) \times P$ be given; our goal is to show that ψ is continuous at $(\gamma_0, \bar{\gamma}_0, p_0)$. We set $X_1 := X \times X \times \mathbb{F}$, $\overline{X}_1 := \overline{X} \times \overline{X} \times \mathbb{F}$, $Z_1 := Z \times Z \times \mathbb{K}$, $E_1 := E \times E$, and $\overline{E}_1 := \overline{E} \times \overline{E}$. Given $(x_0, \bar{x}_0, y_0, \bar{y}_0, t_0) \in (Y \times \overline{M})^{[1]}$, we have $(\sigma(x_0), \sigma^{[1]}(x_0, y_0, t_0), t_0) \in N^{[1]}$ and $(\gamma_0(x_0), \gamma_0^{[1]}(x_0, y_0, t_0), t_0) \in U^{[1]}$. There are

open neighbourhoods $R_1 \subseteq N$ of $\sigma(x_0)$, $R_2 \subseteq Z$ of $\sigma^{[1]}(x_0, y_0, t_0)$, $R_3 \subseteq \mathbb{K}$ of t_0, $V_1 \subseteq U$ of $\gamma_0(x_0)$, and $V_2 \subseteq E$ of $\gamma_0^{[1]}(x_0, y_0, t_0)$ such that

$$R_1 \times R_2 \times R_3 \subseteq N^{[1]} \quad \text{and} \quad V_1 \times V_2 \times R_3 \subseteq U^{[1]}.$$

Then $R_1 \times V_1 \times \overline{E} \times P \times R_2 \times V_2 \times \overline{E} \times \{0\} \times R_3 \subseteq (N \times U \times \overline{E} \times P)^{[1]}$. Abbreviate $N_1 := R_1 \times R_2 \times R_3$ and $U_1 := V_1 \times V_2$. Then

$$\tilde{f}_1 \colon N_1 \times U_1 \times \overline{E}_1 \times P \to F, \quad \tilde{f}_1(x, y, t, u, v, \bar{u}, \bar{v}, p) := \tilde{f}^{[1]}((x, u, \bar{u}, p), (y, v, \bar{v}, 0), t)$$

for $(x, y, t) \in N_1$, $(u, v) \in U_1 = V_1 \times V_2$, $(\bar{u}, \bar{v}) \in \overline{E}_1$, $p \in P$ defines a mapping of class $C^{r+1+k-1}_{\mathbb{K}} = C^{r+k}_{\mathbb{K}}$ on the open subset $N_1 \times U_1 \times \overline{E}_1 \times P$ of $Z_1 \times E_1 \times \overline{E}_1 \times H$. There exists an open neighbourhood $A_1 \subseteq Y$ of x_0 with compact closure $C_1 \subseteq Y$, and open neighbourhoods $A_2 \subseteq X$ of y_0, $\overline{A}_1 \subseteq \overline{M}$ of $\bar{x}_0$, $\overline{A}_2 \subseteq \overline{X}$ of $\bar{y}_0$, and $A_3 \subseteq \mathbb{F} \cap R_3$ of t_0 such that $M_1 := A_1 \times A_2 \times A_3 \subseteq Y^{[1]}$, $\overline{M}_1 := \overline{A}_1 \times \overline{A}_2 \times A_3 \subseteq \overline{M}^{[1]}$, $\sigma(A_1) \subseteq R_1$, $\sigma^{[1]}(A_1 \times A_2 \times A_3) \subseteq R_2$, $\gamma_0(C_1) \subseteq V_1$, and $\gamma_0^{[1]}(A_1 \times A_2 \times A_3) \subseteq V_2$. Let $K_1 \subseteq M_1$ be a compact neighbourhood of (x_0, y_0, t_0), with interior $Y_1 := K_1^0$. Define $\sigma_1 : Y_1 \to N_1$, $\sigma_1(x, y, t) := (\sigma(x), \sigma^{[1]}(x, y, t), t)$ for $x \in A_1$, $y \in A_2$, $t \in A_3$. Then σ_1 is a $C^r_{\mathbb{F}}$-map. We set

$$f_1 := \tilde{f}_1 \circ (\sigma_1 \times \mathrm{id}_{U_1} \times \mathrm{id}_{\overline{E}_1} \times \mathrm{id}_P) \colon Y_1 \times U_1 \times \overline{E}_1 \times P \to F.$$

Abbreviate $z := (x_0, \bar{x}_0, y_0, \bar{y}_0, t_0)$. Then

$$\psi_1 \colon \lfloor K_1, U_1 \rfloor_r \times C^r(\overline{M}_1, \overline{E}_1) \times P \to C^r(Y_1 \times \overline{M}_1, F), \quad \psi_1(\gamma, \bar{\gamma}, p) := f_1(\bullet, p)_*(\gamma \times \bar{\gamma})$$

is a continuous mapping by induction, where $\lfloor K_1, U_1 \rfloor_r \subseteq C^r(M_1, E_1)$. Let $B_1 \subseteq A_1$, $B_2 \subseteq A_2$ and $B_3 \subseteq A_3$ be open neighbourhoods of x_0, y_0 and t_0, respectively, such that $B_1 \times B_2 \times B_3 \subseteq Y_1$. Then $Q_z := B_1 \times \overline{A}_1 \times B_2 \times \overline{A}_2 \times B_3$ is an open neighbourhood of z in $(Y \times \overline{M})^{[1]}$. We define $\delta \colon Q_z \to Y_1 \times \overline{M}_1$, $\delta(x, \bar{x}, y, \bar{y}, t) := (x, y, t, \bar{x}, \bar{y}, t)$. Since ψ_1 is continuous, also

$$\psi_2 := C^r(\delta, F) \circ \psi_1 \colon \lfloor K_1, U_1 \rfloor_r \times C^r(\overline{M}_1, \overline{E}_1) \times P \to C^r(Q_z, F)$$

is continuous (by Lemma 4.4). Note that

$$\Omega := \{\gamma \in \lfloor K, U \rfloor_{r+1} \cap \lfloor C_1, V_1 \rfloor_{r+1} \colon \gamma^{[1]}|_{M_1} \in \lfloor K_1, V_2 \rfloor_r\}$$

is an open neighbourhood of γ_0 in $\lfloor K, U \rfloor_{r+1}$ (cf. Remark 4.2 (b), Lemma 4.4 and Lemma 4.22). The linear maps

$$\pi \colon C^{r+1}(M, E) \to C^r(M_1, E), \quad \pi(\gamma)(x, y, t) := \gamma(x) \quad \text{and}$$

$$\bar{\pi} \colon C^{r+1}(\overline{M}, \overline{E}) \to C^r(\overline{M}_1, \overline{E}), \quad \bar{\pi}(\bar{\gamma})(\bar{x}, \bar{y}, t) := \bar{\gamma}(\bar{x})$$

are continuous (see Remark 4.2 (a) and Lemma 4.4), and we have $\pi(\Omega) \subseteq \lfloor K_1, V_1 \rfloor_r$. Also

$$C^{r+1}(M, E) \to C^r(M_1, E), \quad \gamma \mapsto \gamma^{[1]}|_{M_1} \quad \text{and}$$

$$C^{r+1}(\overline{M},\overline{E})\to C^r(\overline{M}_1,\overline{E}),\qquad \bar\gamma\mapsto \bar\gamma^{[1]}|_{\overline{M}_1}$$

are continuous linear mappings (see Remark 4.2 and Lemma 4.4), and the first of these maps Ω into $\lfloor K_1,V_2\rfloor_r$. Let $\rho_z : C^r((Y\times\overline{M})^{[1]},F)\to C^r(Q_z,F)$, $\rho_z(\eta):=\eta|_{Q_z}$ be the restriction map. In view of (51) and the definition of ψ_2, we have

$$\psi(\gamma,\bar\gamma,p)(x,\bar x,y,\bar y,t)=\psi_2\left((\pi(\gamma),\gamma^{[1]}|_{M_1}),\,(\bar\pi(\bar\gamma),\bar\gamma^{[1]}|_{\overline{M}_1}),\,p\right)(x,\bar x,y,\bar y,t)$$

for $(\gamma,\bar\gamma,p)\in\Omega\times C^{r+1}(\overline{M},\overline{E})\times P$, $(x,\bar x,y,\bar y,t)\in Q_z$, showing that

$$\rho_z\circ\psi|_{\Omega\times C^{r+1}(\overline{M},\overline{E})\times P}$$

is continuous on $\Omega\times C^{r+1}(\overline{M},\overline{E})\times P$, which is a neighbourhood of $(\gamma_0,\bar\gamma_0,p_0)$. Thus, we have achieved the following: given any $z=(x_0,\bar x_0,y_0,\bar y_0,t_0)\in(Y\times\overline{M})^{[1]}$, we have found an open neighbourhood Q_z of z in $(Y\times\overline{M})^{[1]}$ such that $\rho_z\circ\psi$ is continuous at $(\gamma_0,\bar\gamma_0,p_0)$. In view of Lemma 4.6, this entails that ψ is continuous at $(\gamma_0,\bar\gamma_0,p_0)$, as desired.

Induction step on k.

Suppose the assertion of the lemma is correct for some $k\in\mathbb{N}_0$ and all $r\in\mathbb{N}_0$. Let σ and $\tilde f$ be given which satisfy the hypotheses of the lemma when k is replaced with $k+1$. Then $\phi:\lfloor K,U\rfloor_r\times C^r(\overline{M},\overline{E})\times P\to C^r(Y\times\overline{M},F)$ is of class $C^k_{\mathbb{K}}$ (and thus continuous), by induction. For all $(\gamma,\bar\gamma,p,\eta,\bar\eta,q,t)\in(\lfloor K,U\rfloor_r\times C^r(\overline{M},\overline{E})\times P)^{]1[}\subseteq\lfloor K,U\rfloor_r\times C^r(\overline{M},\overline{E})\times P\times C^r(M,E)\times C^r(\overline{M},\overline{E})\times H\times\mathbb{K}$, we calculate

$$\begin{aligned}
&\tfrac{1}{t}(\phi(\gamma+t\eta,\bar\gamma+t\bar\eta,p+tq)-\phi(\gamma,\bar\gamma,p))\,(x,\bar x)\\
&\quad=\ \tfrac{1}{t}(\tilde f(\sigma(x),\gamma(x)+t\eta(x),\bar\gamma(\bar x)+t\bar\eta(\bar x),p+tq)-\tilde f(\sigma(x),\gamma(x),\bar\gamma(\bar x),p))\\
&\quad=\ \tilde f^{[1]}((\sigma(x),\gamma(x),\bar\gamma(\bar x),p),\ (0,\eta(x),\bar\eta(\bar x),q),\ t)
\end{aligned}\tag{52}$$

for all $x\in Y$ and $\bar x\in\overline{M}$. On the open subset

$$W\subseteq N\times U\times E\times\overline{E}\times\overline{E}\times P\times H\times\mathbb{K}$$

consisting of those $(x,y,z,\bar y,\bar z,p,q,t)$ such that $(y,z,t)\in U^{[1]}$ and $(p,q,t)\in P^{[1]}$, we define a $C^{r+k+1-1}_{\mathbb{K}}=C^{r+k}_{\mathbb{K}}$-map

$$\tilde h\colon W\to F,\quad \tilde h(x,y,z,\bar y,\bar z,p,q,t):=\tilde f^{[1]}((x,y,\bar y,p),\ (0,z,\bar z,q),\ t)\,.$$

Motivated by (52), we consider the map

$$\begin{gathered}
\psi\colon(\lfloor K,U\rfloor_r\times C^r(\overline{M},\overline{E})\times P)^{[1]}\to C^r(Y\times\overline{M},F)\,,\\
\psi((\gamma,\bar\gamma,p),\ (\eta,\bar\eta,q),\ t)(x,\bar x):=\tilde h(\sigma(x),\gamma(x),\eta(x),\bar\gamma(\bar x),\bar\eta(\bar x),p,q,t)
\end{gathered}$$

which extends $\phi^{]1[}$. If we can show that ψ is $C^k_{\mathbb{K}}$, then ψ is continuous and thus $\psi=\phi^{[1]}$. Thus ϕ will be of class $C^1_{\mathbb{K}}$, with $\phi^{[1]}=\psi$ of class $C^k_{\mathbb{K}}$, entailing that ϕ is of class $C^{k+1}_{\mathbb{K}}$ (see **1.7**), as required.

Let $\mathcal{A}$ be the set of all relatively compact, non-empty, open subsets of Y. As a consequence of Lemmas 1.15, 1.16, 4.6 and 4.12, the mapping ψ will be of class $C^k_{\mathbb{K}}$ if we can show that $\rho_Q \circ \psi$ is of class $C^k_{\mathbb{K}}$ for all $Q \in \mathcal{A}$, where $\rho_Q \colon C^r(Y \times \overline{M}, F) \to C^r(Q \times \overline{M}, F)$, $\rho_Q(\gamma) := \gamma|_{Q\times\overline{M}}$. Here $\rho_Q \circ \psi$ will be of class $C^k_{\mathbb{K}}$ if we can show that every $(\gamma_0, \bar{\gamma}_0, p_0, \eta_0, \bar{\eta}_0, q_0, t_0) \in (\lfloor K, U\rfloor_r \times C^r(\overline{M}, \overline{E}) \times P)^{[1]}$ has an open neighbourhood T such that $(\rho_Q \circ \psi)|_T$ is of class $C^k_{\mathbb{K}}$ (see Lemma 1.12).

Case $t_0 \neq 0$: Then we can take $T := (\lfloor K, U\rfloor_r \times C^r(\overline{M}, \overline{E}) \times P)^{]1[}$. In fact, $\psi|_T = \phi^{]1[}$ (and thus $\rho_Q \circ \psi|_T$) is of class $C^k_{\mathbb{K}}$ since so is ϕ.

Case $t_0 = 0$: In view of the compactness of $\gamma_0(\overline{Q}) \subseteq U$ and $\eta_0(\overline{Q}) \subseteq E$, where $\overline{Q}$ denotes the closure of Q in Y, there exist open neighbourhoods B_1 of $\gamma_0(\overline{Q})$ in U, B_2 of $\eta_0(\overline{Q})$ in E and an open zero-neighbourhood $B_3 \subseteq \mathbb{K}$ such that $B_1 + B_3 \cdot B_2 \subseteq U$ and thus $B_1 \times B_2 \times B_3 \subseteq U^{[1]}$. Shrinking B_3 if necessary, we find open neighbourhoods B_4 of p_0 in P and B_5 of q_0 in H such that $B_4 + B_3 \cdot B_5 \subseteq P$ and thus $B_4 \times B_5 \times B_3 \subseteq P^{[1]}$. Define $U_1 := B_1 \times B_2 \subseteq U \times E$ and $P_1 := B_4 \times B_5 \times B_3 \subseteq P \times H \times \mathbb{K}$. Then $N \times U_1 \times \overline{E} \times \overline{E} \times P_1 \subseteq W$, and $\tilde{g} := \tilde{h}|_{N\times U_1 \times \overline{E}^2 \times P_1} \colon N \times U_1 \times \overline{E}^2 \times P_1 \to F$ is a mapping of class $C^{r+k}_{\mathbb{K}}$. We define

$$g := \tilde{g} \circ (\sigma|_Q \times \mathrm{id}_{U_1} \times \mathrm{id}_{\overline{E}^2} \times \mathrm{id}_{P_1}) \colon Q \times U_1 \times \overline{E}^2 \times P_1 \to F\,.$$

By induction hypothesis, the mapping

$$\Phi \colon \lfloor \overline{Q}, U_1\rfloor_r \times C^r(\overline{M}, \overline{E}^2) \times P_1 \to C^r(Q \times \overline{M}, F), \quad \Phi(\gamma, \bar{\gamma}, p) := g(\bullet, p)_*(\gamma \times \bar{\gamma})$$

is of class $C^k_{\mathbb{K}}$, where $\lfloor \overline{Q}, U_1\rfloor_r \subseteq C^r(M, E \times E)$. Note that

$$\begin{gathered}\theta \colon (C^r(M,E) \times C^r(\overline{M}, \overline{E}) \times H)^2 \times \mathbb{K} \to C^r(M, E\times E) \times C^r(\overline{M}, \overline{E} \times \overline{E}) \times H \times H \times \mathbb{K},\\ (\gamma, \bar{\gamma}, p, \eta, \bar{\eta}, q, t) \mapsto ((\gamma, \eta), (\bar{\gamma}, \bar{\eta}), p, q, t)\end{gathered}$$

is an isomorphism of topological $\mathbb{K}$-vector spaces such that $(\gamma_0, \bar{\gamma}_0, p_0, \eta_0, \bar{\eta}_0, q_0, t_0) \in T := \theta^{-1}(\lfloor \overline{Q}, U_1\rfloor_r \times C^r(\overline{M}, \overline{E}^2) \times P_1) \cap (\lfloor K, U\rfloor_r \times C^r(\overline{M}, \overline{E}) \times P)^{[1]}$. To complete the proof, it only remains to observe that $\rho_Q \circ \psi|_T = \Phi \circ \theta|_T^{\theta(T)}$ is a mapping of class $C^k_{\mathbb{K}}$.

C Proof of Proposition 11.3

In the situation of Proposition 11.3, let H be a finite-dimensional $\mathbb{K}$-vector space, and $P \subseteq H$ be open. If we can prove the following lemma, then apparently Proposition 11.3 will follow:

Lemma C.1 *The mapping* $\Theta \colon C^{r+k}(F \times P, E) \times C^r(M, F) \times P \to C^r(M, E)$, $\Theta(\gamma, \eta, p) := \gamma(\bullet, p) \circ \eta$ *is of class* $C^k_{\mathbb{K}}$*. If* $k \geq 1$*, then*

$$\begin{aligned}&\Theta^{[1]}((\gamma, \eta, p),\, (\gamma_1, \eta_1, p_1),\, t)\\ &\qquad = \gamma^{[1]}((\bullet, p),\, (\bullet, p_1),\, t) \circ (\eta, \eta_1) + \gamma_1(\bullet, p + tp_1) \circ (\eta + t\eta_1) \qquad (53)\end{aligned}$$

for all $((\gamma, \eta, p),\, (\gamma_1, \eta_1, p_1),\, t) \in (C^{r+k}(F \times P, E) \times C^r(M, F) \times P)^{[1]}$.

Proof. As in the proof of Lemma 11.4, we may assume that $k, r \in \mathbb{N}_0$. The proof is by induction on k.

The case $k = 0$

We proceed by induction on r. If $r = 0$, then a trivial variation of the argument used in the proof of Lemma 11.4 shows that Θ is continuous.

Induction step on r. Let $r \in \mathbb{N}$, and suppose that the lemma holds for $k = 0$, when r is replaced with $r - 1$. It then suffices to show continuity of Θ in the case where M is an open subset of its modeling space Z. In fact, if M is a $C^r_{\mathbb{K}}$-manifold, Lemma 4.12 entails that Θ will be continuous if, for any chart $\kappa\colon U \to V \subseteq Z$ of M, the map

$$h\colon C^r(F \times P, E) \times C^r(M, F) \times P \to C^r(V, E), \quad h(\gamma, \eta, p) := \Theta(\gamma, \eta, p) \circ \kappa^{-1}$$

is continuous. But

$$h(\gamma, \eta, p) \;=\; \gamma({\scriptstyle\bullet}, p) \circ (\eta \circ \kappa^{-1}) \;=\; \Xi(\gamma, \eta \circ \kappa^{-1}, p)\,, \tag{54}$$

where $\Xi\colon C^r(F \times P, E) \times C^r(V, F) \times P \to C^r(V, E)$, $\Xi(\gamma, \sigma, p) := \gamma({\scriptstyle\bullet}, p) \circ \sigma$. Recall that the pullback $C^r(M, F) \to C^r(V, F)$, $\eta \mapsto \eta \circ \kappa^{-1}$ is continuous linear (Lemma 4.11). Thus (54) shows that h will be continuous if Ξ is continuous. Since V is open in Z, this completes the reduction step to the case where M is open in Z.

To complete the induction step on r in the case $k = 0$, by the preceding we may assume now that M is an open subset of Z. The map $\Theta\colon C^r(F \times P, E) \times C^r(M, F) \times P \to C^r(M, E)$ is continuous as a map into $C(M, E)$, by the case $r = 0$ already settled. Hence, in view of Remark 4.2 (b), Θ will be continuous if we can show that the map

$$C^r(F \times P, E) \times C^r(M, F) \times P \to C^{r-1}(M^{[1]}, E), \quad (\gamma, \eta, p) \mapsto \Theta(\gamma, \eta, p)^{[1]}$$

is continuous, where

$$\Theta(\gamma, \eta, p)^{[1]}(x, y, t) = \gamma^{[1]}((\eta(x), p),\, (\eta^{[1]}(x, y, t), 0),\, t) \tag{55}$$

for all $(x, y, t) \in M^{[1]}$, by the Chain Rule. By (55), we have

$$\Theta(\gamma, \eta, p)^{[1]} = \widetilde{\Theta}(\gamma^{[1]} \circ \rho,\, (\eta \circ \mathrm{pr}_1, \eta^{[1]}, \mathrm{pr}_3),\, p) \tag{56}$$

for all (γ, η, p) in the domain of Θ, where $\mathrm{pr}_1\colon M^{[1]} \to M$, $(x, y, t) \mapsto x$ and $\mathrm{pr}_3\colon M^{[1]} \to \mathbb{K}$, $(x, y, t) \mapsto t$ are the coordinate projections,

$$\rho\colon F^{[1]} \times P \to (F \times P)^{[1]}, \quad \rho((u, v, t),\, p) := ((u, p),\, (v, 0),\, t)$$

is continuous linear, and $\widetilde{\Theta}\colon C^{r-1}(F^{[1]} \times P, E) \times C^{r-1}(M^{[1]}, F^{[1]}) \times P \to C^{r-1}(M^{[1]}, E)$,

$$\widetilde{\Theta}(\sigma, \tau, p) := \sigma({\scriptstyle\bullet}, p) \circ \tau$$

is continuous by induction. Because the mapping $C^r(M, F) \to C^{r-1}(M^{[1]}, F)$, $\eta \mapsto \eta^{[1]}$ and both of the pullbacks $C^r(M, F) \to C^{r-1}(M^{[1]}, F)$, $\eta \mapsto \eta \circ \mathrm{pr}_1$ and $C^{r-1}(\rho, E)$: $C^{r-1}((F \times P)^{[1]}, E) \to C^{r-1}(F^{[1]} \times P, E)$ are continuous, we deduce from (56) that $(\gamma, \eta, p) \mapsto \Theta(\gamma, \eta, p)^{[1]}$ is continuous. Hence so is Θ.

Induction step on k

Let $k \in \mathbb{N}$, and suppose that the assertion of the lemma holds when k is replaced with $k-1$, for all integers $r \in \mathbb{N}_0$. Let $r \in \mathbb{N}_0$. Given an element $((\gamma, \eta, p), (\gamma_1, \eta_1, p_1), t) \in (C^{r+k}(F \times P, E) \times C^r(M, F) \times P)^{]1[}$, we calculate for $x \in M$:

$$\begin{aligned} &\tfrac{1}{t}\left(\Theta(\gamma + t\gamma_1, \eta + t\eta_1, p + tp_1) - \Theta(\gamma, \eta, p)\right)(x) \\ &\quad = \tfrac{1}{t}\Big(\gamma(\eta(x) + t\eta_1(x), p + tp_1) - \gamma(\eta(x), p)\Big) + \gamma_1(\eta(x) + t\eta_1(x), p + tp_1) \\ &\quad = \gamma^{[1]}((\eta(x), p), (\eta_1(x), p_1), t) + \gamma_1(\eta(x) + t\eta_1(x), p + tp_1). \end{aligned} \tag{57}$$

Thus $\Theta^{]1[}$ coincides with the restriction to $(C^{r+k}(F \times P, E) \times C^r(M, F) \times P)^{]1[}$ of the mapping $g \colon (C^{r+k}(F \times P, E) \times C^r(M, F) \times P)^{[1]} \to C^r(M, F)$,

$$g((\gamma, \eta, p), (\gamma_1, \eta_1, p_1), t) := \widetilde{\Theta}(\gamma^{[1]} \circ \rho, (\eta, \eta_1), (p, p_1, t)) + \Theta(\gamma_1, \eta + t\eta_1, p + tp_1), \tag{58}$$

where $\rho \colon F^2 \times P^{[1]} \to (F \times P)^{[1]}$, $\rho(u, v, p, p_1, t) := ((u, p), (v, p_1), t)$ is continuous linear and $\widetilde{\Theta} \colon C^{r+k-1}(F^2 \times P^{[1]}, E) \times C^r(M, F^2) \times P^{[1]} \to C^r(M, E)$,

$$\widetilde{\Theta}(\sigma, \tau, (p, p_1, t)) := \sigma(\bullet, (p, p_1, t)) \circ \tau$$

is of class $C^{k-1}_{\mathbb{K}}$, by induction. Since Θ is $C^{k-1}_{\mathbb{K}}$ and hence continuous as a consequence of the induction hypothesis, in order that Θ be $C^k_{\mathbb{K}}$, it only remains to show that g is of class $C^{k-1}_{\mathbb{K}}$ (then $\Theta^{[1]} = g$, which also establishes (53)). Now, the second summand in (58) clearly describes a $C^{k-1}_{\mathbb{K}}$-map. Also the first summand is $C^{k-1}_{\mathbb{K}}$, as pullbacks are continuous linear and $\widetilde{\Theta}$ is $C^{k-1}_{\mathbb{K}}$. $\square$

D Smoothness vs. weak smoothness over local fields

We vary a classical result of A. Grothendieck concerning mappings on open sets in $\mathbb{R}^n$:

Theorem D.1 *Let $\mathbb{K}$ be a local field, E and F be topological $\mathbb{K}$-vector spaces, $f \colon U \to F$ a map on an open subset of E, and $k \in \mathbb{N}_0$. If E is metrizable and F is Mackey complete and locally convex, then we have:*

(a) *If f is C^k, then f is weakly C^k, viz. $\lambda \circ f$ is C^k for each $\lambda \in F'$.*

(b) *If f is weakly C^{k+1}, then f is C^k.*

In particular, f is smooth if and only if f is weakly smooth.

Proof. (a) is a trivial consequence of the Chain Rule.

(b) If we can prove (b) in the special case where $U = E$ is finite-dimensional, then for general f the composition $f \circ \gamma$ will be weakly C^{k+1} and thus C^k, for every smooth

map $\gamma\colon \mathbb{K}^{k+1} \to U$. Hence f will be C^k by [3, Thm. 12.4]. We may therefore assume that $U = E = \mathbb{K}^\ell$ for some ℓ. The proof is by induction on k.

If $k = 0$ and $f\colon E = \mathbb{K}^\ell \to F$ is weakly C^1, let $x \in U$ and $W \subseteq F$ be a 0-neighbourhood. Define

$$B \;:=\; \{f^{]1[}(x,y,t)\colon y \in K, t \in \mathbb{O} \setminus \{0\}\}\,,$$

where $\mathbb{O} \subseteq \mathbb{K}$ is the valuation ring, K a compact 0-neighbourhood in E, and $f^{]1[}(x,y,t) := t^{-1}(f(x+ty) - f(x))$. Then $\lambda(B) \subseteq (\lambda \circ f)^{[1]}(\{x\} \times K \times \mathbb{O})$ is compact and thus bounded, for each $\lambda \in F'$, whence B is bounded in F by [77, Thm. 4.21]. Thus there is $t \in \mathbb{O} \setminus \{0\}$ such that $tB \subseteq W$. Then $f(y) - f(x) \in tB \subseteq W$ for every $y \in x + tK$. Hence f is continuous at x.

Induction step. If $k \geq 1$ and $f\colon E = \mathbb{K}^\ell \to F$ is weakly C^{k+1}, given $x, y \in E$ choose a sequence $(t_n)_{n \in \mathbb{N}}$ of pairwise distinct elements in $\mathbb{O} \setminus \{0\}$ such that $t_n \to 0$. Set

$$B \;:=\; \{(t_n - t_m)^{-1}(f^{]1[}(x,y,t_m) - f^{]1[}(x,y,t_n)) : n, m \in \mathbb{N}\}\,.$$

Then $\lambda(B) = \{(\lambda \circ f)^{[2]}((x,y,t_n),\,(0,0,1),\,t_m - t_n)\colon n, m \in \mathbb{N}\}$ is contained in the compact set $(\lambda \circ f)^{[2]}(\{x\} \times \{y\} \times \mathbb{O} \times \{(0,0,1)\} \times \mathbb{O})$ and thus bounded, for each $\lambda \in F'$, whence B is bounded by [77, Thm. 4.21]. Since $f^{]1[}(x,y,t_m) - f^{]1[}(x,y,t_n) \in (t_m - t_n)B$, we deduce that $(f^{]1[}(x,y,t_n))_{n \in \mathbb{N}}$ is a Mackey-Cauchy sequence in F and thus convergent; we let $g(x,y,0)$ be its limit. Then $\lambda(g(x,y,0)) = \lim_{n\to\infty} (\lambda \circ f)^{]1[}(x,y,t_n) = (\lambda \circ f)^{[1]}(x,y,0)$ for each λ. Furthermore, trivially $\lambda(g(x,y,t)) = (\lambda \circ f)^{[1]}(x,y,t)$ for $g(x,y,t) := t^{-1}(f(x+ty) - f(x))$ for all $(x,y,t) \in E \times E \times \mathbb{K}^\times$. Thus $\lambda \circ g = (\lambda \circ f)^{[1]}$ is C^k for each λ, whence g is C^{k-1} (and thus C^0), by induction. Hence f is C^1 with $f^{[1]} = g$ of class C^{k-1}, whence f is C^k. □

E Towards a *p*-adic analogue of Boman's Theorem

Boman's Theorem asserts that a mapping $f : \mathbb{R}^n \to \mathbb{R}$ is smooth if and only if $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is smooth for each smooth curve $\gamma\colon \mathbb{R} \to \mathbb{R}$ (see [9, Thm. 1]). As is well known, this implies that a mapping $f : U \to F$ from an open subset $U \subseteq E$ of a metrizable real locally convex space E to a real locally convex space F is smooth if and only if $f \circ \gamma$ is smooth for each smooth curve $\gamma : \mathbb{R} \to U$ (cf. [47, Thm. 12.8]). It is natural to ask whether versions of Boman's theorem remain valid over arbitrary locally compact topological fields, or at least in the p-adic case. A positive answer to this question (which is still open) would be quite useful.[25] In the present section, we show that, as in the real locally convex case, the validity of Boman's Theorem for functions $\mathbb{K}^2 \to \mathbb{K}$ entails its validity for functions between open subsets of suitable topological vector spaces. Our main tools are the exponential laws from Section 12, the adaptations of Grothendieck's Theorem from Appendix D, and the characterization of smooth maps on metrizable topological vector spaces from [3, Thm. 12.4].

[25]For example, combining the results of this section and [26, Cor. 4.3], this would entail that the exponential map of a smooth (not necessarily analytic) p-adic Banach-Lie group is automatically a C^∞-diffeomorphism. This would be an important step towards proving the conjecture that every smooth p-adic Banach-Lie group admits a smoothly compatible p-adic analytic Lie group structure.

Lemma E.1 *Let $\mathbb{K}$ be the field of real numbers or an ultrametric field. Let $\mathcal{A}$ be one of the following classes of topological $\mathbb{K}$-vector spaces:*

- *The class of all topological $\mathbb{K}$-vector spaces;*
- *The class of all sequentially complete topological $\mathbb{K}$-vector spaces;*
- *The class of all Mackey complete topological $\mathbb{K}$-vector spaces;*
- *The class of all locally convex topological $\mathbb{K}$-vector spaces;*
- *The class of all sequentially complete, locally convex topological $\mathbb{K}$-vector spaces; or:*
- *The class of all Mackey complete, locally convex topological $\mathbb{K}$-vector spaces.*

Suppose that Boman's Theorem holds for mappings from $\mathbb{K}^2$ to topological vector spaces in $\mathcal{A}$, i.e., assume the validity of the following statement:

> *If $f : \mathbb{K}^2 \to F$ is a map into a topological $\mathbb{K}$-vector space $F \in \mathcal{A}$, and $f \circ \gamma \colon \mathbb{K} \to F$ is $C^\infty_\mathbb{K}$ for each $C^\infty_\mathbb{K}$-curve $\gamma \colon \mathbb{K} \to \mathbb{K}^2$, then f is of class $C^\infty_\mathbb{K}$.*

Then Boman's Theorem holds for mappings $f \colon U \to F$ from open subsets $U \subseteq E$ of metrizable topological $\mathbb{K}$-vector spaces E to topological $\mathbb{K}$-vector spaces $F \in \mathcal{A}$, i.e., smoothness of $f \circ \gamma \colon \mathbb{K} \to F$ for each smooth curve $\gamma \colon \mathbb{K} \to U$ entails smoothness of f.

Proof. Assuming that the hypothesis of the lemma is correct, we first show by induction on $n \in \mathbb{N}$ that Boman's Theorem holds for mappings $f : \mathbb{K}^n \to F$, for all $F \in \mathcal{A}$. The case $n = 1$ is trivial, and the case $n = 2$ holds by the hypothesis of the lemma. Thus, assume that $n \geq 2$ now, assume that Boman's Theorem holds for functions on $\mathbb{K}^n$, and assume that $f : \mathbb{K}^{n+1} \to F$ is a function into some $F \in \mathcal{A}$ such that $f \circ \gamma$ is smooth for each smooth curve $\gamma : \mathbb{K} \to \mathbb{K}^{n+1}$. Then $f(x, \bullet) : \mathbb{K}^{n-1} \to F$ is smooth along smooth curves, for any $x \in \mathbb{K}^2$, and thus $f(x, \bullet)$ is smooth, by induction. Therefore $f^\vee : \mathbb{K}^2 \to C^\infty(\mathbb{K}^{n-1}, F)$, $f^\vee(x) := f(x, \bullet)$ is correctly defined. By Proposition 12.6 (b), the map $f = (f^\vee)^\wedge$ will be smooth if $f^\vee$ is smooth. Since $C^\infty(\mathbb{K}^{n-1}, F) \in \mathcal{A}$ by Proposition 4.19, $f^\vee$ will be smooth if $f^\vee \circ \gamma \colon \mathbb{K} \to C^\infty(\mathbb{K}^{n-1}, F)$ is smooth for each smooth curve $\gamma = (\gamma_1, \gamma_2) \colon \mathbb{K} \to \mathbb{K}^2$, by the hypothesis of the lemma. By Proposition 12.6 (b), $f^\vee \circ \gamma$ will be smooth if $g := (f^\vee \circ \gamma)^\wedge \colon \mathbb{K}^n \to F$ is smooth. Note that $g(t, y) = f(\gamma_1(t), \gamma_2(t), y)$ for $t \in \mathbb{K}$, $y \in \mathbb{K}^{n-1}$, and thus $(g \circ \eta)(t) = f(\gamma_1(\eta_1(t)), \gamma_2(\eta_1(t)), \eta_2(t), \ldots, \eta_n(t))$ is smooth for each smooth curve $\eta = (\eta_1, \ldots, \eta_n) \colon \mathbb{K} \to \mathbb{K}^n$, as f is smooth along smooth curves. Being smooth along smooth curves, $g \colon \mathbb{K}^n \to F$ is smooth, by the induction hypothesis. In view of our reduction steps, this means that also f is smooth. This finishes the induction.

To complete the proof, let U be an open subset of a metrizable topological $\mathbb{K}$-vector space, $F \in \mathcal{A}$ and $f : U \to F$ be a mapping which is smooth along smooth curves. Then the composition $f \circ \eta \colon \mathbb{K}^n \to F$ is smooth along smooth curves and hence smooth by what has already been shown, for any $n \in \mathbb{N}$ and smooth map $\eta \colon \mathbb{K}^n \to U$. Hence f is smooth, by [3, Thm. 12.4]. $\square$

Theorem E.2 *Let $\mathbb{K}$ be a local field. Suppose that Boman's Theorem holds for functions from $\mathbb{K}^2$ to $\mathbb{K}$, i.e., assume the validity of the following statement:*

> *If $f : \mathbb{K}^2 \to \mathbb{K}$ is a function such that $f \circ \gamma : \mathbb{K} \to \mathbb{K}$ is $C^\infty_{\mathbb{K}}$ for each $C^\infty_{\mathbb{K}}$-curve $\gamma : \mathbb{K} \to \mathbb{K}^2$, then f is of class $C^\infty_{\mathbb{K}}$.*

Then Boman's theorem holds for mappings $f : U \to F$ from open subsets $U \subseteq E$ of metrizable topological $\mathbb{K}$-vector spaces E to Mackey complete, locally convex topological $\mathbb{K}$-vector spaces F, i.e., smoothness of $f \circ \gamma \colon \mathbb{K} \to F$ for each smooth curve $\gamma \colon \mathbb{K} \to U$ entails smoothness of f.

Proof. Let $\mathcal{A}$ be the class of Mackey complete, locally convex topological $\mathbb{K}$-vector spaces. We only need to show that the present hypothesis entails the hypothesis of Lemma E.1. To this end, let $f \colon \mathbb{K}^2 \to F$ be a map into a Mackey complete, locally convex topological $\mathbb{K}$-vector space which is smooth along each smooth curve. If $\lambda \colon F \to \mathbb{K}$ is a continuous linear functional, then $\lambda \circ f : \mathbb{K}^2 \to \mathbb{K}$ is smooth along each smooth curve and hence smooth, by the hypothesis of the present theorem. Thus f is weakly smooth and hence smooth, by our analogue of Grothendieck's Theorem (Theorem D.1). Thus, the hypothesis of Lemma E.1 is verified. □

In the real case, we do have Boman's Theorem available. Thus, we arrive at the conclusion:

Proposition E.3 *Let E be a metrizable real topological vector space, $U \subseteq E$ be an open subset and F a locally convex real topological vector space. Then a mapping $f \colon U \to F$ is smooth if and only if $f \circ \gamma$ is smooth for each smooth curve $\gamma \colon \mathbb{R} \to U$.*

Proof. We only need to verify the hypothesis of Lemma E.1 for $\mathbb{K} = \mathbb{R}$ and the class $\mathcal{A}$ of all locally convex, real topological vector spaces. Thus, suppose that $f \colon \mathbb{R}^2 \to F$ is a map into a real locally convex space which is smooth along smooth curves. Then $\lambda \circ f \colon \mathbb{R}^2 \to \mathbb{R}$ is smooth along smooth curves and hence smooth by Boman's Theorem [9, Thm. 1], for each continuous linear functional λ on the completion $\overline{F}$ of F. Hence $f : \mathbb{R}^2 \to \overline{F}$ is weakly smooth and hence smooth, by Grothendieck's classical theorem. For any $n \in \mathbb{N}$ and $x, y \in \mathbb{R}^2$, we have $d^n f(x, y, \ldots, y) = \frac{d^n}{dt^n}\big|_{t=0} f(x + ty) \in F$, as f is smooth along the smooth curve $\mathbb{R} \to \mathbb{R}^2$, $t \mapsto x + ty$. Hence $d^n f(x, \bullet)$ has image in F, by polarization (cf. [7, Thm. A]). Now, $f : \mathbb{R}^2 \to F$ being a smooth map into $\overline{F}$ with $\mathrm{im}(d^n f) \subseteq F$ for each $n \in \mathbb{N}_0$, we deduce that f is smooth as a map into F, as required. □

Remark E.4 Note that E need not be locally convex in Proposition E.3, and that no completeness properties whatever are presumed for E nor F. Therefore, the result is slightly more general than the results in the literature (and those in the folklore).

F Spaces of sections in vector bundles and mappings between them

In this appendix, we define bundles of topological vector spaces, topologize their spaces of sections, and study differentiability properties of mappings between open subsets of spaces of sections.

In the following, $\mathbb{K}$ denotes an arbitrary topological field. Additional properties of $\mathbb{K}$ (local compactness) will be spelt out explicitly where they are needed. Unless specified otherwise, $r \in \mathbb{N}_0 \cup \{\infty\}$.

Definition F.1 Let M be a $C^r_{\mathbb{K}}$-manifold, modeled on a topological $\mathbb{K}$-vector space Z, and F be a topological $\mathbb{K}$-vector space. A $C^r_{\mathbb{K}}$-*vector bundle* over M, with typical fibre F, is a $C^r_{\mathbb{K}}$-manifold E, together with a $C^r_{\mathbb{K}}$-surjection $\pi\colon E \to M$ and equipped with a $\mathbb{K}$-vector space structure on each fibre $E_x := \pi^{-1}(\{x\})$, such that for each $x_0 \in M$, there exists an open neighbourhood M_ψ of x_0 in M and a $C^r_{\mathbb{K}}$-diffeomorphism

$$\psi\colon \pi^{-1}(M_\psi) \to M_\psi \times F$$

(called a "local trivialization of E about x_0") such that $\psi(E_x) = \{x\} \times F$ for each $x \in M_\psi$ and $\mathrm{pr}_F \circ \psi|_{E_x} : E_x \to F$ is $\mathbb{K}$-linear (and thus an isomorphism of topological $\mathbb{K}$-vector spaces with respect to the topology on E_x induced by E), where $\mathrm{pr}_F\colon M_\psi \times F \to F$ is the projection on the second coordinate.

Remark F.2 In the situation of Definition F.1, suppose we are given two local trivializations $\psi : \pi^{-1}(M_\psi) \to M_\psi \times F$ and $\phi : \pi^{-1}(M_\phi) \to M_\phi \times F$. Then $\phi(\psi^{-1}(x,v)) = (x, g_{\phi,\psi}(x).v)$ for some function $g_{\phi,\psi}\colon M_\phi \cap M_\psi \to \mathrm{GL}(F) \subseteq L(F,F)$ (the space of continuous linear self-maps), and $G_{\phi,\psi} : (M_\phi \cap M_\psi) \times F \to F$, $(x,v) \mapsto g_{\phi,\psi}(x).v$ is a $C^r_{\mathbb{K}}$-map (since $\phi \circ \psi^{-1}$ is so).

Definition F.3 A $C^r_{\mathbb{K}}$-*section* of a $C^r_{\mathbb{K}}$-vector bundle $\pi\colon E \to M$ is a $C^r_{\mathbb{K}}$-map $\sigma\colon M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$. Its *support* $\mathrm{supp}(\sigma)$ is the closure of $\{x \in M : \sigma(x) \neq 0_x\}$. We let $C^r(M,E)$ be the set of all $C^r_{\mathbb{K}}$-sections in E. If $\mathbb{K}$ is locally compact and M is finite-dimensional, we let $C^r_K(M,E)$ be the set of all $C^r_{\mathbb{K}}$-sections with support contained in a given compact subset $K \subseteq M$.

Making use of scalar multiplication and addition in the individual fibres, we obtain natural vector space structures on $C^r(M,E)$ and $C^r_K(M,E)$. The zero-element is the *zero-section* $0_\bullet\colon M \to E$, $x \mapsto 0_x \in E_x$.

Definition F.4 If $\pi\colon E \to M$ is a $C^r_{\mathbb{K}}$-vector bundle with typical fibre F, $\sigma\colon M \to E$ a $C^r_{\mathbb{K}}$-section, and $\psi\colon \pi^{-1}(M_\psi) \to M_\psi \times F$ a local trivialization, we define $\sigma_\psi := \mathrm{pr}_F \circ \psi \circ \sigma|_{M_\psi}\colon M_\psi \to F$. Thus $\psi(\sigma(x)) = (x, \sigma_\psi(x))$ for all $x \in M_\psi$.

Note that σ_ψ is a mapping of class $C^r_{\mathbb{K}}$ here. The symbols $g_{\phi,\psi}$, $G_{\phi,\psi}$, and σ_ψ will always be used with the meanings just described, without further explanation.

Definition F.5 If $\pi\colon E \to M$ is a vector bundle and $\mathcal{A}$ a set of local trivializations ψ of E whose domains cover E, then we call $\mathcal{A}$ an *atlas* of local trivializations.

Lemma F.6 *If $\pi\colon E \to M$ is a $C^r_{\mathbb{K}}$-vector bundle with typical fibre F, and $\mathcal{A}$ an atlas of local trivializations for E, then*

$$\Gamma\colon C^r(M,E) \to \prod_{\psi\in\mathcal{A}} C^r(M_\psi, F), \qquad \sigma \mapsto (\sigma_\psi)_{\psi\in\mathcal{A}}$$

is an injection, whose image is the closed vector subspace

$$H := \big\{(f_\psi) \in \prod_{\psi\in\mathcal{A}} C^r(M_\psi, F)\colon\ (\forall \phi,\psi \in \mathcal{A},\ \forall x \in M_\phi \cap M_\psi)\ f_\phi(x) = g_{\phi,\psi}(x).f_\psi(x)\,\big\}$$

of $\prod_{\psi\in\mathcal{A}} C^r(M_\psi, F)$.

Proof. It is obvious that $\operatorname{im}\Gamma \subseteq H$, and clearly Γ is injective. If now $(f_\psi)_{\psi\in\mathcal{A}} \in H$, we define $\sigma : M \to E$ via $\sigma(x) := \psi^{-1}(x, f_\psi(x))$ if $x \in M_\psi$. By definition of H, $\sigma(x)$ is independent of the choice of ψ. As $\psi \circ \sigma|_{M_\psi} = (\mathrm{id}_{M_\psi}, f_\psi)$, the mapping $\sigma : M \to E$ is of class $C^r_{\mathbb{K}}$. Thus σ is a $C^r_{\mathbb{K}}$-section, and $\Gamma(\sigma) = (f_\psi)_{\psi\in\mathcal{A}}$ by definition of σ. We deduce that $\operatorname{im}\Gamma = H$. The closedness of H follows from the continuity of the point evaluations $C^r(M_\psi, F) \to F$, $\gamma \mapsto \gamma(x)$ for $x \in M_\psi$. $\square$

Definition F.7 Let $\pi : E \to M$ be a $C^r_{\mathbb{K}}$-vector bundle, with typical fibre F, and $\mathcal{A}$ be the atlas of all local trivializations of E. We give $C^r(M,E)$ the vector topology making the linear mapping

$$\Gamma\colon C^r(M,E) \to \prod_{\psi\in\mathcal{A}} C^r(M_\psi, F), \quad \sigma \mapsto (\sigma_\psi)_{\psi\in\mathcal{A}}$$

a topological embedding.

By the preceding definition, the topology on $C^r(M,E)$ is initial with respect to the family $(\theta_\psi)_{\psi\in\mathcal{A}}$, where $\theta_\psi\colon C^r(M,E) \to C^r(M_\psi, F)$, $\sigma \mapsto \sigma_\psi$.

Remark F.8 In the situation of Definition F.7, assume that $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$ or an ultrametric field, and assume that F is locally convex. Then $C^r(M_\psi, F)$ is locally convex for each $\psi \in \mathcal{A}$, by Proposition 4.19 (b). Hence also $C^r(M,E)$ is locally convex.

It suffices to work with any atlas of local trivializations.

Lemma F.9 *The topology on $C^r(M,E)$ described in Definition* F.7 *is initial with respect to $(\theta_\psi)_{\psi\in\mathcal{B}}$, for any atlas $\mathcal{B} \subseteq \mathcal{A}$ of local trivializations for E.*

Proof. We let $\mathcal{O}$ be the initial topology on $C^r(M,E)$ with respect to $(\theta_\psi)_{\psi\in\mathcal{B}}$, which apparently is coarser than initial topology with respect to $(\theta_\psi)_{\psi\in\mathcal{A}}$. Fix a local trivialization $\phi\in\mathcal{A}$. Then $\{M_\phi\cap M_\psi : \psi\in\mathcal{B}\}$ is an open cover for M_ϕ. In view of Lemma 4.12, the map θ_ϕ will be continuous on $(C^r(M,E),\mathcal{O})$ if the map $(C^r(M,E),\mathcal{O})\to C^r(M_\phi\cap M_\psi,F)$, $\sigma\mapsto\theta_\phi(\sigma)|_{M_\phi\cap M_\psi}$ is continuous for all $\psi\in\mathcal{B}$. But, with $G_{\phi,\psi}$ as in Remark F.2, the latter mapping is the composition of $(G_{\phi,\psi})_*\colon C^r(M_\phi\cap M_\psi,F)\to C^r(M_\phi\cap M_\psi,F)$ (which is continuous by Proposition 4.16) and $C^r(M,E)\to C^r(M_\phi\cap M_\psi,F)$, $\sigma\mapsto\theta_\psi(\sigma)|_{M_\phi\cap M_\psi}$, which is continuous by Lemma 4.11. Thus, θ_ϕ being continuous on $(C^r(M,E),\mathcal{O})$ for each $\phi\in\mathcal{A}$, the topology $\mathcal{O}$ is finer than the initial topology with respect to the family $(\theta_\phi)_{\phi\in\mathcal{A}}$, which completes the proof. □

Definition F.10 Suppose that $\pi_1\colon E_1\to M$ and $\pi_2\colon E_2\to M$ are $C^r_{\mathbb{K}}$-vector bundles over the same base, with typical fibres F_1 and F_2, respectively. A mapping $f\colon U\to E_2$, defined on an open subset U of E_1, is called a *bundle map* if it preserves fibres, *i.e.*, $f(U\cap(E_1)_x)\subseteq(E_2)_x$ for all $x\in M$. Then, given local trivializations $\psi:\pi_1^{-1}(M_\psi)\to M_\psi\times F_1$ and $\phi\colon\pi_2^{-1}(M_\phi)\to M_\phi\times F_2$ of E_1, resp., E_2, we have

$$\phi(f(\psi^{-1}(x,v)))=(x,f_{\phi,\psi}(x,v))$$

for all $(x,v)\in U_{\phi,\psi}:=\psi(U\cap E_1|_{M_\psi\cap M_\phi})\subseteq(M_\psi\cap M_\phi)\times F_1$, for a uniquely determined mapping

$$f_{\phi,\psi}\colon U_{\phi,\psi}\to F_2\,.$$

Theorem F.11 *Let $r,k\in\mathbb{N}_0\cup\{\infty\}$, M be a (not necessarily finite-dimensional) $C^{r+k}_{\mathbb{K}}$-manifold and $\pi_1:E_1\to M$, $\pi_2:E_2\to M$ be $C^{r+k}_{\mathbb{K}}$-vector bundles over M, whose typical fibres F_1, resp., F_2 are arbitrary topological $\mathbb{K}$-vector spaces. Then the following holds:*

(a) *If H is a topological $\mathbb{K}$-vector space, $P\subseteq H$ an open subset and $f\colon E_1\times P\to E_2$ a $C^{r+k}_{\mathbb{K}}$-map such that $f_p:=f(\bullet,p)\colon E_1\to E_2$ is a bundle map, for each $p\in P$, then*

$$\Theta:\,C^r(M,E_1)\times P\to C^r(M,E_2)\,,\quad \Theta(\sigma,p)\,:=\,C^r(M,f_p)(\sigma)\,=\,f(\bullet,p)\circ\sigma$$

is a $C^k_{\mathbb{K}}$-map.

(b) *If $f\colon E_1\to E_2$ is a bundle map of class $C^{r+k}_{\mathbb{K}}$, then*

$$C^r(M,f):\,C^r(M,E_1)\to C^r(M,E_2),\quad \sigma\mapsto f\circ\sigma$$

is a mapping of class $C^k_{\mathbb{K}}$.

Proof. (b) directly follows from (a) by taking $P:=H:=\{0\}$. It thus suffices to prove (a). We let $(U_j)_{j\in J}$ be an open cover of M such that for every $j\in J$, there are local trivializations $\psi_j:\pi_1^{-1}(U_j)\to U_j\times F_1$ and $\phi_j:\pi_2^{-1}(U_j)\to U_j\times F_2$. For each $j\in J$, the mapping $h_j\colon U_j\times F_1\times P\to F_2$, $h_j(x,y,p):=(f_p)_{\phi_j,\psi_j}(x,y)$ is $C^{r+k}_{\mathbb{K}}$. We have

$$\beta_j(\Theta(\sigma))\,=\,((f_p)_{\phi_j,\psi_j})_*(\alpha_j(\sigma))\,=\,h_j(\bullet,p)_*(\alpha_j(\sigma))$$

for all $p \in P$ and $\sigma \in C^r(M, E_1)$, where $\alpha_j : C^r(M, E_1) \to C^r(U_j, F_1)$, $\sigma \mapsto \sigma_{\psi_j}$ and $\beta_j : C^r(M, E_2) \to C^r(U_j, F_2)$, $\sigma \mapsto \sigma_{\phi_j}$ are continuous linear maps. By Proposition 4.16, the map $C^r(U_j, F_1) \times P \to C^r(U_j, F_2)$, $(\gamma, p) \mapsto h(\bullet, p)_*(\gamma)$ is $C^k_{\mathbb{K}}$. In view of Lemma F.6 and Lemma F.9, Lemma 1.15 shows that Θ is of class $C^k_{\mathbb{K}}$. $\square$

The most interesting cases of Theorem F.11 are (i) $r = k = \infty$; (ii) $r \in \mathbb{N}_0 \cup \{\infty\}$, $k = 0$.

To illustrate the results, we show that spaces of sections are topological modules over the corresponding function algebras. Recall the notion of (Whitney) sums of vector bundles:

F.12 If $\pi_j : E_j \to M$ are $C^r_{\mathbb{K}}$-vector bundles with fibre F_j for $j \in \{1, 2\}$, over the same base M, then $E_1 \oplus E_2 := \bigcup_{x \in M} (E_1)_x \times (E_2)_x$ is a C^r-vector bundle over M in a natural way, with projection $\pi : E_1 \oplus E_2 \to M$, $v \mapsto x$ if $v \in (E_1)_x \times (E_2)_x$. Let $\phi_j : \pi_j^{-1}(M_{\phi_j}) \to M_{\phi_j} \times F_j$ be local trivializations of E_j for $j \in \{1, 2\}$, and $M_{\phi_1 \oplus \phi_2} := M_{\phi_1} \cap M_{\phi_2}$. Then

$$\phi_1 \oplus \phi_2 : \pi^{-1}(M_{\phi_1 \oplus \phi_2}) \to M_{\phi_1 \oplus \phi_2} \times (F_1 \times F_2),$$

$(\phi_1 \oplus \phi_2)(v, w) := (x, \mathrm{pr}_{F_1}(\phi_1(v)), \mathrm{pr}_{F_2}(\phi_2(w)))$ for $(v, w) \in (E_1)_x \times (E_2)_x$, is a local trivialization of $E_1 \oplus E_2$. It is easy to see that the linear mapping

$$C^r(M, E_1) \times C^r(M, E_2) \to C^r(M, E_1 \oplus E_2), \quad (\sigma_1, \sigma_2) \mapsto (x \mapsto (\sigma_1(x), \sigma_2(x)))$$

is an isomorphism of topological $\mathbb{K}$-vector spaces.

As an immediate consequence of Theorem F.11, we have:

Corollary F.13 *Let $\mathbb{K}$ be a topological field and $\pi : E \to M$ be a $C^r_{\mathbb{K}}$-vector bundle, whose fibre is a topological $\mathbb{K}$-vector space F. Then $C^r(M, E)$ is a topological $C^r(M, \mathbb{K})$-module.*

Proof. The function space $C^r(M, \mathbb{K})$ can be identified with the space $C^r(M, M \times \mathbb{K})$ of $C^r_{\mathbb{K}}$-sections of the trivial bundle $\mathrm{pr}_1 : M \times \mathbb{K} \to M$ with fibre $\mathbb{K}$ (cf. Lemma F.9). Thus $C^r(M, \mathbb{K}) \times C^r(M, E) \cong C^r(M, (M \times \mathbb{K}) \oplus E)$. Using this identification, the multiplication map $C^r(M, \mathbb{K}) \times C^r(M, E) \to C^r(M, E)$ has the form

$$C^r(M, \mu) : C^r(M, (M \times \mathbb{K}) \oplus E) \to C^r(M, E),$$

where $\mu : (M \times \mathbb{K}) \oplus E \to E$ is the bundle map defined via $\mu((x, z), v) := zv \in E_x$ (scalar multiplication) for all $x \in M$, $z \in \mathbb{K}$, and $v \in E_x$. Given any local trivialization $\psi : \pi^{-1}(M_\psi) \to M_\psi \times F$ of E, using the global trivialization $\phi := \mathrm{id} : M \times \mathbb{K} \to M \times \mathbb{K}$ we have $\mu_{\psi, \phi \oplus \psi}(x, z, v) = zv \in F$, for all $(x, z, v) \in M_\psi \times \mathbb{K} \times F$, showing that the map $\mu_{\psi, \phi \oplus \psi} : M_\psi \times \mathbb{K} \times F \to F$ is of class $C^r_{\mathbb{K}}$. Thus μ is a $C^r_{\mathbb{K}}$-bundle map. By Theorem F.11 (b) (applied with $k = 0$), $C^r(M, \mu)$ is continuous. $\square$

If $\pi : E \to M$ is a $C^r_{\mathbb{K}}$-vector bundle and U an open subset of M, then $\pi|_{E|_U}^U : E|_U \to U$ makes the open submanifold $E|_U := \pi^{-1}(U)$ of E a $C^r_{\mathbb{K}}$-vector bundle over the base U.

Definition F.14 Let $\mathbb{K}$ be a locally compact topological field, $\pi\colon E \to M$ be a $C^r_{\mathbb{K}}$-vector bundle over a finite-dimensional $C^r_{\mathbb{K}}$-manifold M, and $K \subseteq M$ be compact. We equip $C^r_K(M,E)$ with the topology induced by $C^r(M,E)$.

Lemma F.15 *Let $\pi : E \to M$ be a $C^r_{\mathbb{K}}$-vector bundle over a finite-dimensional $C^r_{\mathbb{K}}$-manifold M, and U be an open subset of M. Then the following holds:*

(a) *The restriction map $C^r(M,E) \to C^r(U,E|_U)$, $\sigma \mapsto \sigma|_U$ is continuous.*

(b) *Assume that $\mathbb{K}$ is locally compact, M a finite-dimensional $C^r_{\mathbb{K}}$-manifold and $K \subseteq U$ a compact subset. Then the restriction map $\rho_U : C^r_K(M,E) \to C^r_K(U,E|_U)$ is an isomorphism of topological vector spaces.*

Proof. (a) Since every local trivialization of $E|_U$ also is a local trivialization of E, Part (a) is apparent from the definition of the topologies.

(b) As a consequence of (a), also ρ_U is continuous. Apparently, it is a linear bijection. To see that ρ_U is an isomorphism of topological vector spaces, we let $\mathcal{A}_0$ be an atlas of local trivializations for $E|_{M\setminus K}$, and $\mathcal{A}_1$ be an atlas for $E|_U$. Then the topology on $C^r_K(U,E|_U)$ is initial with respect to the family of mappings $\theta^U_\psi\colon C^r_K(U,E|_U) \to C^r(M_\psi,F)$, $\sigma \mapsto \sigma_\psi$, for $\psi \in \mathcal{A}_1$. Furthermore, by Lemma F.9, the topology on $C^r_K(M,E)$ is initial with respect to family of mappings $\theta^M_\psi : C^r_K(M,E) \to C^r(M_\psi,F)$ for $\psi \in \mathcal{A}_0 \cup \mathcal{A}_1$ (defined analogously). As $\theta^U_\psi \circ \rho_U = \theta^M_\psi$ for all $\psi \in \mathcal{A}_1$ and $\theta^M_\psi = 0$ for all $\psi \in \mathcal{A}_0$, the assertion easily follows. $\square$

A variant of Lemma 4.12 (and Lemma F.6) will be needed.

Lemma F.16 *Let $\pi\colon E \to M$ be a $C^r_{\mathbb{K}}$-vector bundle, and $(U_i)_{i\in I}$ be an open cover of M. For each $i \in I$, let $\rho_i\colon C^r(M,E) \to C^r(U_i,E|_{U_i})$ be the restriction map. Then*

$$\rho := (\rho_i)_{i\in I} :\ C^r(M,E) \to \prod_{i\in I} C^r(U_i,E|_{U_i})\,, \quad \rho(\sigma) := (\sigma|_{U_i})_{i\in I}$$

is a topological embedding, with closed image.

Proof. Each ρ_i being continuous by Lemma F.15 (a), also ρ is continuous. To see that ρ is an embedding, consider the set $\mathcal{A}$ of all local trivializations $\psi\colon E|_{M_\psi} \to M_\psi \times F$ of E such $M_\psi \subseteq U_i$ for some $i \in I$; here F is the typical fibre of E. Then $\mathcal{A}$ is an atlas of local trivializations for E, and the topology on $C^r(M,E)$ is initial with respect to the family $(\theta_\psi)_{\psi\in\mathcal{A}}$ of the maps $\theta_\psi\colon C^r(M,E) \to C^r(M_\psi,F)$, $\theta_\psi(\sigma) := \sigma_\psi$ (Lemma F.9). If $M_\psi \subseteq U_i$, then $\theta_\psi(\sigma) = \sigma_\psi = (\sigma|_{U_i})_\psi = (\rho_i(\sigma))_\psi$ shows that θ_ψ is continuous with respect to the topology induced by ρ on $C^r(M,E)$. As a consequence, ρ is a topological embedding. To complete the proof, note that

$$H := \Big\{(\sigma_i)_{i\in I} \in \prod_{i\in I} C^r(U_i,E|_{U_i})\colon (\forall i,j \in I)\ \sigma_i|_{U_i\cap U_j} = \sigma_j|_{U_i\cap U_j}\Big\}$$

is closed in $\prod_{i\in I} C^r(U_i, E|_{U_i})$, the restriction maps $C^r(U_i, E|_{U_i}) \to C^r(U_i \cap U_j, E|_{U_i\cap U_j})$ being continuous (Lemma F.15 (a)). Clearly $\operatorname{im}(\rho) \subseteq H$. If, conversely, $(\sigma_i)_{i\in I} \in H$, then $\sigma(x) := \sigma_i(x)$ if $x \in U_i$ gives a well-defined section $\sigma\colon M \to E$. Clearly $\sigma \in C^r(M, E)$ and $\rho(\sigma) = (\sigma_i)_{i\in I}$. Thus $\operatorname{im}(\rho) = H$ is closed. □

Definition F.17 Let $\mathbb{K}$ be a locally compact topological field, M be a paracompact $C^r_{\mathbb{K}}$-manifold modeled on a finite-dimensional $\mathbb{K}$-vector space Z, and $\pi\colon E \to M$ be a $C^r_{\mathbb{K}}$-vector bundle with fibre an arbitrary (Hausdorff, not necessarily locally convex) topological $\mathbb{K}$-vector space F. Then the set

$$C^r_c(M, E) := \{\sigma \in C^r(M, E)\colon\ \operatorname{supp}(\sigma) \text{ is compact}\,\}$$

of compactly supported $C^r_{\mathbb{K}}$-sections is a $\mathbb{K}$-vector subspace of $C^r(M, E)$, and $C^r_c(M, E) = \bigcup_{K\in\mathcal{K}(M)} C^r_K(M, E)$, where $\mathcal{K}(M)$ denotes the set of all compact subsets of M. In the following, we consider three vector topologies on $C^r_c(M, E)$:

(a) We write $C^r_c(M, E)_{\text{tvs}}$ for $C^r_c(M, E)$, equipped with the finest (a priori not necessarily Hausdorff) vector topology making the inclusion maps $\lambda_K\colon C^r_K(M, E) \to C^r_c(M, E)$ continuous for each compact subset $K \subseteq M$. Thus $C^r_c(M, E)_{\text{tvs}} = \varinjlim C^r_K(M, E)$ in the category of not necessarily Hausdorff topological $\mathbb{K}$-vector spaces and continuous $\mathbb{K}$-linear maps.

(b) If F is locally convex, we write $C^r_c(M, E)_{\text{lcx}}$ for $C^r_c(M, E)$, equipped with the finest (a priori not necessarily Hausdorff) locally convex vector topology making the inclusion maps $\lambda_K\colon C^r_K(M, E) \to C^r_c(M, E)$ continuous for each compact subset $K \subseteq M$. Thus $C^r_c(M, E)_{\text{lcx}} = \varinjlim C^r_K(M, E)$ in the category of not necessarily Hausdorff, locally convex topological $\mathbb{K}$-vector spaces and continuous $\mathbb{K}$-linear maps.

(c) As M is paracompact and locally compact, there exists a locally finite cover $\mathcal{U} = (U_i)_{i\in I}$ of M by relatively compact, open subsets $U_i \subseteq M$. We define

$$\rho_{\mathcal{U}}\colon C^r_c(M, E) \to \bigoplus_{i\in I} C^r(U_i, E|_{U_i}), \quad \rho_{\mathcal{U}}(\sigma) := (\rho_i(\sigma))_{i\in I} = (\sigma|_{U_i})_{i\in I}\,,$$

where $\rho_i : C^r_c(M, E) \to C^r(U_i, E|_{U_i})$ is the restriction map for $i \in I$. We write $C^r_c(M, E)_{\text{box}}$ for $C^r_c(M, E)$, equipped with the topology $\mathcal{O}_{\mathcal{U}}$ induced by $\rho_{\mathcal{U}}$, where the direct sum is endowed with the box topology.

Lemma F.18 *In the situation of Definition* F.17 (c)*, assume that both $\mathcal{U} = (U_i)_{i\in I}$ and $\mathcal{V} = (V_j)_{j\in J}$ are locally finite covers of M by relatively compact open sets. Then $\mathcal{O}_{\mathcal{U}} = \mathcal{O}_{\mathcal{V}}$. In other words, the box topology on $C^r_c(M, E)$ is independent of the choice of $\mathcal{U}$.*

Proof. The topologies $\mathcal{O}_{\mathcal{U}}$ and $\mathcal{O}_{\mathcal{V}}$ are induced by $\rho_{\mathcal{U}} : C^r_c(M, E) \to \bigoplus_{i\in I} C^r(U_i, E|_{U_i})$, $\rho_{\mathcal{U}}(\sigma) := (\sigma|_{U_i})_{i\in I}$ and $\rho_{\mathcal{V}} : C^r_c(M, E) \to \bigoplus_{j\in J} C^r(V_j, E|_{V_j})$, $\rho_{\mathcal{V}}(\sigma) := (\sigma|_{V_j})_{j\in J}$, respectively. Using Lemma F.16 instead of Lemma 4.12, we can repeat the proof of Lemma 8.10 verbatim to get the desired result. □

Proposition F.19 *Let M be a paracompact, finite-dimensional $C^r_{\mathbb{K}}$-manifold over a locally compact topological field $\mathbb{K}$. Let $\pi\colon E \to M$ be a $C^r_{\mathbb{K}}$-vector bundle over M, with typical fibre a topological $\mathbb{K}$-vector space F. Then the following holds:*

(a) *The box topology on $C^r_c(M,E)_{\mathrm{box}}$ is Hausdorff. For every locally finite cover $\mathcal{U} = (U_i)_{i\in I}$ of M by relatively compact, open subsets $U_i \subseteq M$, the map*

$$\rho_{\mathcal{U}}\colon C^r_c(M,E)_{\mathrm{box}} \to \bigoplus_{i\in I} C^r(U_i, E|_{U_i}), \quad \rho_{\mathcal{U}}(\sigma) := (\sigma|_{U_i})_{i\in I}$$

has closed image, and $\rho_{\mathcal{U}}|^{\mathrm{im}\,\rho_{\mathcal{U}}}$ is an isomorphism of topological vector spaces. The inclusion map $C^r_c(M,E)_{\mathrm{box}} \to C^r(M,E)$ is continuous. If F is locally convex, then $C^r_c(M,E)_{\mathrm{box}}$ is locally convex.

(b) *The inclusion map $\lambda_K\colon C^r_K(M,E) \to C^r_c(M,E)_{\mathrm{box}}$ is continuous and induces the given topology on $C^r_K(M,E)$, for each compact subset $K \subseteq M$.*

(c) *The map $\Phi\colon C^r_c(M,E)_{\mathrm{tvs}} \to C^r_c(M,E)_{\mathrm{box}}$, $\Phi(\gamma) := \gamma$ is continuous. Thus $C^r_c(M,E)_{\mathrm{tvs}}$ is Hausdorff and induces the given topology on each $C^r_K(M,E)$. If $\mathbb{K} \neq \mathbb{C}$ and M is σ-compact, then Φ is an isomorphism of topological $\mathbb{K}$-vector spaces.*

(d) *If F is locally convex, then $\Psi\colon C^r_c(M,E)_{\mathrm{lcx}} \to C^r_c(M,E)_{\mathrm{box}}$, $\Psi(\gamma) := \gamma$ is continuous. Hence $C^r_c(M,E)_{\mathrm{lcx}}$ is Hausdorff and induces the given topology on each $C^r_K(M,E)$. If $\mathbb{K} \neq \mathbb{C}$ and M is σ-compact, then Ψ is an isomorphism of topological $\mathbb{K}$-vector spaces.*

(e) *If $\mathbb{K}$ is a local field and $\mathcal{U} = (U_i)_{i\in I}$ is a cover of M by mutually disjoint, compact open sets (cf. Lemma* 8.3 (b)*), then*

$$\rho_{\mathcal{U}}\colon C^r_c(M,E)_{\mathrm{box}} \to \bigoplus_{i\in I} C^r(U_i, E|_{U_i}), \quad \rho_{\mathcal{U}}(\sigma) := (\sigma|_{U_i})_{i\in I}$$

is an isomorphism of topological vector spaces onto the direct sum, equipped with the box topology.

(f) *If $\mathbb{K}$ is a local field and F is locally convex, then Ψ is an isomorphism of topological $\mathbb{K}$-vector spaces, i.e., $C^r_c(M,E)_{\mathrm{lcx}} = C^r_c(M,E)_{\mathrm{box}}$.*

In particular, $C^r_c(M,E)_{\mathrm{box}} = C^r_c(M,E)_{\mathrm{tvs}} = C^r_c(M,E)_{\mathrm{lcx}}$ if $\mathbb{K} \neq \mathbb{C}$ and M is σ-compact.

Proof. (a) Using Lemma F.16 instead of Lemma 4.12 and Remark F.8 instead of Proposition 4.19 (b), the proof of Proposition 8.13 (a) carries over.

(b) Using Lemma F.15 (b) and Lemma F.16 instead of Lemma 4.24 and Lemma 4.12, respectively, the proof of Proposition 8.13 (b) can be repeated verbatim.

(c) Note that, $C^r(U_i, E|_{U_i})$ being a topological $C^r(U_i, \mathbb{K})$-module (Corollary F.13), the multiplication operator $m_h \colon C^r(U_i, E|_{U_i}) \to C^r(M, E|_{U_i})$, $m_h(\sigma)(x) := h(x)\sigma(x)$ is continuous, for each $i \in I$ and $h \in C^r(U_i, \mathbb{K})$. The assertion therefore follows along the lines of the proof of Proposition 8.13 (c) (with $\mathbb{F} := \mathbb{K}$), using Lemma F.15 (b) instead of Lemma 4.24.

(d) We argue as in the proof of Proposition 8.13 (d).

(e) We argue as in the proof of Proposition 8.13 (e), taking $\mathbb{F} := \mathbb{K}$.

(f) Using Lemma F.15 (b) instead of Lemma 4.24, we can proceed as in the proof of Proposition 8.13 (e) (taking $\mathbb{F} := \mathbb{K}$). □

Throughout the following, spaces of compactly supported sections in vector bundles will always be equipped with the box topology. We abbreviate $C^r_c(M, E) := C^r_c(M, E)_{\text{box}}$.

Remark F.20 If $\mathbb{K} = \mathbb{R}$, M is σ-compact and the fibre F is locally convex, then the box topology on $C^r_c(M, E)$ coincides with the locally convex topology traditionally considered on this space of compactly supported sections, as a consequence of Proposition F.19 (d) and Proposition 4.19 (d).

Remark F.21 Let M be a paracompact, finite-dimensional $C^r_{\mathbb{K}}$-manifold over a locally compact field $\mathbb{K}$ (where $r \in \mathbb{N}_0 \cup \{\infty\}$), and $\pi : E \to M$ be a $C^r_{\mathbb{K}}$-vector bundle, with fibre an arbitrary topological $\mathbb{K}$-vector space. Let $(U_i)_{i \in I}$ be a locally finite cover of M by relatively compact, open subsets $U_i \subseteq M$ and $\rho_i \colon C^r_c(M, E) \to C^r(U_i, E|_{U_i})$, $\rho_i(\sigma) := \sigma|_{U_i}$ be the restriction map for $i \in I$. Then

$$(C^r_c(M, E), (\rho_i)_{i \in I})$$

is a patched topological vector space, by Proposition F.19 (a).

The Ω-Lemma with Parameters

In the following, we prove generalizations of the so-called "Ω-Lemma" (see [57, Thm. 8.7]), formulated in [57] for mappings between subsets of spaces of compactly supported smooth sections in finite-dimensional real vector bundles. An essential ingredient of the proof will be a version of Proposition 8.21 for functions depending on parameters.

Lemma F.22 *Let $\mathbb{K}$ be a locally compact topological field and $\pi \colon E \to M$ be a $C^r_{\mathbb{K}}$-vector bundle over a paracompact, finite-dimensional $C^r_{\mathbb{K}}$-manifold M. Let $\Omega \subseteq E$ be an open subset. Then*

$$C^r_c(M, \Omega) \ := \ \{\sigma \in C^r_c(M, E) \colon \sigma(M) \subseteq \Omega\}$$

is an open (possibly empty) subset of $C^r_c(M, E)$.

Proof. The present proof is not the shortest one; it is stated as follows because in exactly this form it can be re-used to prove Theorem F.23.

Let $\sigma \in C^r_c(M, \Omega)$. Using the paracompactness and local compactness of M, we find

locally finite covers $(U_i)_{i\in I}$ and $(M_i)_{i\in I}$ of M by relatively compact, open sets such that $K_i := \overline{U_i} \subseteq M_i$ and $E|_{M_i}$ is a trivial vector bundle, for each $i \in I$.[26] Let $\psi_i \colon E|_{M_i} \to M_i \times F$ be a trivialization of $E|_{M_i}$, where F is the typical fibre of E. Then

$$\kappa :\ C^r_c(M,E) \to \bigoplus_{i\in I} C^r(M_i,F)\,, \quad \kappa(\tau) := (\tau_{\psi_i})_{i\in I} = \big((\tau|_{M_i})_{\psi_i}\big)_{i\in I}$$

is a topological embedding, by definition of the box topology on $C^r_c(M,E)$ and Lemma F.9. For each $i \in I$, the set $\Omega_i := \psi_i(\Omega \cap E|_{M_i})$ is an open neighbourhood of the compact subset $\{(x, \sigma_{\psi_i}(x)) \colon x \in K_i\}$ of $M_i \times F$. Since σ_{ψ_i} is continuous, a standard compactness argument provides finite families $(U_{i,j})_{j\in J_i}$ and $(M_{i,j})_{j\in J_i}$ of relatively compact, open subsets $M_{i,j} \subseteq M_i$ and relatively compact, open subsets $U_{i,j} \subseteq M_{i,j}$ such that $K_i \subseteq \bigcup_{j\in J_i} U_{i,j}$, and open 0-neighbourhoods $W_{i,j} \subseteq F$ such that

$$M_{i,j} \times (\sigma_{\psi_i}(M_{i,j}) + W_{i,j}) \subseteq \Omega_i \qquad \text{for each } i \in I \text{ and } j \in J_i\,. \tag{59}$$

Set $L := \{(i,j) \colon i \in I,\ j \in J_i\}$. After replacing I by L, $(U_i)_{i\in I}$ by $(U_{i,j})_{(i,j)\in L}$, $(M_i)_{i\in I}$ by $(M_{i,j})_{(i,j)\in L}$, and $(\psi_i)_{i\in I}$ by $(\psi_i|_{E|_{M_{i,j}}})_{(i,j)\in L}$, instead of (59) we may assume without loss of generality that there exist open 0-neighbourhoods $W_i \subseteq F$ such that

$$M_i \times (\sigma_{\psi_i}(M_i) + W_i) \subseteq \Omega_i \tag{60}$$

for each $i \in I$. Then $V_i := \sigma_{\psi_i}(K_i) + W_i$ is an open neighbourhood of $\sigma_{\psi_i}(K_i)$ in F, for each $i \in I$, and it is a 0-neighbourhood in F for all but finitely many i. Then $\lfloor K_i, V_i \rfloor_r := \{\gamma \in C^r(M_i,F) \colon \gamma(K_i) \subseteq V_i\}$ is an open subset of $C^r(M_i,F)$ and

$$V \ := \ \kappa^{-1}\left(\bigoplus_{i\in I} \lfloor K_i, V_i \rfloor_r\right)$$

is an open neighbourhood of σ in $C^r_c(M,E)$. Since $(K_i)_{i\in I}$ is a cover of M, we deduce from (60) that $V \subseteq C^r_c(M,\Omega)$. Thus $C^r_c(M,\Omega)$ is a neighbourhood of σ. □

Theorem F.23 (Ω-Lemma with Parameters) *Let $\mathbb{K}$ be a locally compact topological field and $r,k \in \mathbb{N}_0 \cup \{\infty\}$. Let $\pi_1 \colon E_1 \to M$ and $\pi_2 \colon E_2 \to M$ be $C^{r+k}_{\mathbb{K}}$-vector bundles over the same paracompact, finite-dimensional $C^{r+k}_{\mathbb{K}}$-manifold M, whose fibres F_1, resp., F_2 are arbitrary topological $\mathbb{K}$-vector spaces. Let $P \subseteq Z$ be an open subset of a finite-dimensional $\mathbb{K}$-vector space Z, $\Omega \subseteq E_1$ be an open subset, and*

$$f \colon \Omega \times P \to E_2$$

[26] Every $x \in M$ has a relatively compact open neighbourhood Q_x such that $E|_{Q_x}$ is trivial; let P_x be an open neighbourhood of x such that $\overline{P_x} \subseteq Q_x$. Since M is paracompact, there exists a locally finite open cover $(U_i)_{i\in I}$ subordinate to $(P_x)_{x\in M}$. Then $U_i \subseteq P_{x_i}$ for some $x_i \in M$, whence U_i is relatively compact. By Lemma 8.5, there exists a locally finite cover $(\widetilde{U}_i)_{i\in I}$ of M by relatively compact open sets such that $\overline{U_i} \subseteq \widetilde{U}_i$ for each i. We define $M_i := \widetilde{U}_i \cap Q_{x_i}$; then $(U_i)_{i\in I}$ and $(M_i)_{i\in I}$ have the desired properties.

be a mapping of class $C^{r+k}_{\mathbb{K}}$ such that $f_p := f(\bullet, p)\colon \Omega \to E_2$ is a bundle map, for each $p \in P$. We assume that there exists a compact subset $K \subseteq M$ such that $0_x \in \Omega$ for each $x \in M \setminus K$, and $f(0_x, p) = 0_x$ for each $x \in M \setminus K$ and $p \in P$ (using the same symbol $0_\bullet$ for the 0-section of E_1 and E_2, resp.) Then the mapping

$$\phi\colon C^r_c(M, \Omega) \times P \to C^r_c(M, E_2)\,, \quad \phi(\sigma, p) := f_p \circ \sigma$$

is $C^k_{\mathbb{K}}$. In particular, if $k = r = \infty$, then $\phi\colon C^\infty_c(M, \Omega) \times P \to C^\infty_c(M, E_2)$ is smooth.

Proof. Note first that for $\sigma \in C^r_c(M, \Omega)$ and $p \in P$, we have

$$\mathrm{supp}(\phi(\sigma, p)) \;\subseteq\; K \cup \mathrm{supp}(\sigma)\,.$$

Hence $\phi(\sigma, p)$ is indeed compactly supported, and thus $\mathrm{im}(\phi) \subseteq C^r_c(M, E_2)$. To see that ϕ is of class $C^k_{\mathbb{K}}$, we now fix $\sigma \in C^r_c(M, \Omega)$. We let $(U_i)_{i\in I}$, $(M_i)_{i\in I}$, $(\psi_i)_{i\in I}$, $(K_i)_{i\in I}$, $(V_i)_{i\in I}$, $\kappa:\ C^r_c(M, E_1) \to \bigoplus_{i\in I} C^r(M_i, F_1)$, $\kappa(\tau) := (\tau_{\psi_i})_{i\in I}$ and $V := \kappa^{-1}\left(\bigoplus_{i\in I} \lfloor K_i, V_i \rfloor_r\right)$ be as in the proof of Lemma F.22 (applied with $E := E_1$ and $F := F_1$). Because we can choose each Q_x so small that also $E_2|_{Q_x}$ is trivial in the proof of Lemma F.22, we may assume without loss of generality that also $E_2|_{M_i}$ is trivial, for each $i \in I$. Let $\phi_i\colon E_2|_{M_i} \to M_i \times F_2$ be a trivialization. Abbreviate $\omega_i := \phi_i|^{U_i \times F_2}_{E_2|_{U_i}}\colon E_2|_{U_i} \to U_i \times F_2$. Then

$$\kappa_2:\ C^r_c(M, E_2) \to \bigoplus_{i\in I} C^r(U_i, F_2)\,, \quad \kappa_2(\tau) := (\tau_{\omega_i})_{i\in I} = \big((\tau|_{U_i})_{\omega_i}\big)_{i\in I}$$

is a topological embedding onto a closed vector subspace, as a consequence of Proposition F.19 (a) and Lemma F.9. For each $i \in I$, the map

$$f_i:\ M_i \times V_i \times P \to F_2\,, \quad f_i(x, y, p) \;:=\; \mathrm{pr}_2(\phi_i(f(\psi_i^{-1}(x, y), p)))$$

is of class $C^{r+k}_{\mathbb{K}}$, where $\mathrm{pr}_2\colon M_i \times F_2 \to F_2$ is the second coordinate projection. Hence, by Proposition 4.23 (a), the map

$$g_i:\ \lfloor K_i, V_i \rfloor_r \times P \to C^r(U_i, F_2)\,, \quad g_i(\gamma, p)(x) \;:=\; f_i(x, \gamma(x), p)$$

is of class $C^k_{\mathbb{K}}$, where $\lfloor K_i, V_i \rfloor_r := \{\gamma \in C^r(M_i, F_1) : \gamma(K_i) \subseteq V_i\}$. Furthermore, $0 \in \lfloor K_i, V_i \rfloor_r$ and $g_i(0, p) = 0$ whenever $M_i \cap (K \cup \mathrm{supp}(\sigma)) = \emptyset$, which is the case for all but finitely many $i \in I$. Hence, by Proposition 6.10, the map

$$g:\ \left(\bigoplus_{i\in I} \lfloor K_i, V_i \rfloor_r\right) \times P \to \bigoplus_{i\in I} C^r(U_i, F_2)\,, \quad g\Big(\sum_{i\in I} \gamma_i,\, p\Big) \;:=\; \sum_{i\in I} g_i(\gamma_i, p)$$

is of class $C^k_{\mathbb{K}}$. Since the diagram

$$\begin{array}{ccccc} C^r_c(M, \Omega) \times P & \supseteq & V \times P & \xrightarrow{\kappa|_V \times \mathrm{id}_P} & \left(\bigoplus_{i\in I} \lfloor K_i, V_i \rfloor_r\right) \times P \\ & & \phi|_{V\times P} \downarrow & & \downarrow g \\ & & C^r_c(M, E_2) & \xrightarrow{\kappa_2} & \bigoplus_{i\in I} C^r(U_i, F_2) \end{array}$$

commutes and κ_2 is an embedding of topological vector spaces with closed image, we deduce with Lemma 1.15 that $\phi|_{V\times P}$ is $C^k_{\mathbb{K}}$ on the open neighbourhood $V\times P$ of $\{\sigma\}\times P$ in $C^r_c(M,\Omega)\times P$. As σ was arbitrary, ϕ is $C^k_{\mathbb{K}}$. □

Specializing to a singleton set of parameters, we obtain:

Corollary F.24 (Ω-Lemma) *Let $\mathbb{K}$ be a locally compact topological field and $r,k\in\mathbb{N}_0\cup\{\infty\}$. Let $\pi_1: E_1\to M$ and $\pi_2: E_2\to M$ be $C^{r+k}_{\mathbb{K}}$-vector bundles over the same paracompact, finite-dimensional $C^{r+k}_{\mathbb{K}}$-manifold M, whose fibres F_1, resp., F_2 are arbitrary topological $\mathbb{K}$-vector spaces. Let $\Omega\subseteq E_1$ be an open neighbourhood of the image of a section $\sigma_0\in C^r_c(M,E_1)$, and $f:\Omega\to E_2$ be a bundle map of class $C^{r+k}_{\mathbb{K}}$, such that $f\circ\sigma_0$ has compact support. Then the map*

$$\phi\colon C^r_c(M,\Omega)\to C^r_c(M,E_2)\,,\quad \phi(\sigma):=f\circ\sigma$$

is of class $C^k_{\mathbb{K}}$. □

Remark F.25 The following special cases of Corollary F.24 are of particular interest:

(a) If $k=r=\infty$, then $C^\infty_c(M,f)\colon C^\infty_c(M,\Omega)\to C^\infty_c(M,E_2)$, $\sigma\mapsto f\circ\sigma$ is smooth.

(b) If $r\in\mathbb{N}_0$ and $k=0$, then $C^r_c(M,f)\colon C^r_c(M,\Omega)\to C^r_c(M,E_2)$, $\sigma\mapsto f\circ\sigma$ is continuous.

To illustrate the results, let us prove that spaces of compactly supported sections are topological modules over the corresponding test function algebras. First, we observe:

F.26 If $\mathbb{K}$ is locally compact and the manifold M is finite-dimensional and paracompact in the situation of **F.12**, then apparently also the linear mapping

$$C^r_c(M,E_1)\times C^r_c(M,E_2)\to C^r_c(M,E_1\oplus E_2),\quad (\sigma_1,\sigma_2)\mapsto (x\mapsto(\sigma_1(x),\sigma_2(x)))$$

is an isomorphism of topological $\mathbb{K}$-vector spaces.

As an immediate consequence of Theorem F.11 and Corollary F.24, we now obtain:

Corollary F.27 *Let $\mathbb{K}$ be a locally compact topological field, M be a paracompact, finite-dimensional $C^r_{\mathbb{K}}$-manifold, and $\pi\colon E\to M$ be a $C^r_{\mathbb{K}}$-vector bundle, whose fibre is a topological $\mathbb{K}$-vector space F. Then $C^r_c(M,E)$ is a topological $C^r_c(M,\mathbb{K})$-module.*

Proof. The proof of Corollary F.13 carries over verbatim, using Corollary F.24 and **F.26** instead of Theorem F.11 (b) and **F.12**. □

Remark F.28 If A is an associative topological $\mathbb{K}$-algebra and M a $C^r_{\mathbb{K}}$-manifold, we define a *bundle of topological A-modules* as a $C^r_{\mathbb{K}}$-vector bundle $\pi\colon E\to M$ whose typical fibre F is a topological A-module, and equipped with an atlas $\mathcal{A}$ of local trivializations such that $\mathrm{im}(g_{\phi,\psi})$ consists of topological A-module automorphisms of F, for all $\phi,\psi\in\mathcal{A}$. In this case, we see as in the proof of Corollary F.13 that $C^r(M,E)$ is a topological $C^r(M,A)$-module (under pointwise operations). If $\mathbb{K}$ is locally compact and M is finite-dimensional and paracompact, then $C^r_c(M,E)$ is a topological $C^r_c(M,A)$-module (cf. proof of Corollary F.27).

Almost local mappings between spaces of compactly supported sections

Our considerations from Section 10 carry over directly to the case of mappings between spaces of compactly supported sections in vector bundles (equipped with the box topology). Compare the earlier works [32] and [33] for a discussion of the real locally convex case, based on the locally convex direct limit topology on spaces of compactly supported sections.

Definition F.29 Let $\mathbb{K}$ be the field of real numbers or a local field. Given $r, s, k \in \mathbb{N}_0 \cup \{\infty\}$, let $\pi_1 \colon E_1 \to M$ be a $C^r_{\mathbb{K}}$-vector bundle over a paracompact, finite-dimensional $C^r_{\mathbb{K}}$-manifold M, with fibre an arbitrary topological $\mathbb{K}$-vector space F_1. Let $\pi_2 \colon E_2 \to N$ be a $C^s_{\mathbb{K}}$-vector bundle over a paracompact, finite-dimensional $C^s_{\mathbb{K}}$-manifold N, with fibre an arbitrary topological $\mathbb{K}$-vector space F_2. Finally, let $f \colon P \to C^s_c(N, E_2)$ be a mapping, defined on an open subset $P \subseteq C^r_c(M, E_1)$.

(a) f is called *almost local* if there exist locally finite covers $(U_i)_{i\in I}$ of M and $(V_i)_{i\in I}$ of N by relatively compact, open sets such that, for all $i \in I$ and $\sigma, \tau \in P$ with $\sigma|_{U_i} = \tau|_{U_i}$, we have $f(\sigma)|_{V_i} = f(\tau)|_{V_i}$.

(b) f is called *locally almost local* if every $\sigma \in P$ has an open neighbourhood $Q \subseteq P$ such that $f|_Q$ is almost local.

(c) In the special case where $M = N$, we call $f \colon P \to C^s_c(M, E_2)$ a *local* mapping if, for all $x \in M$ and $\sigma \in P$, the element $f(\sigma)(x)$ only depends on the germ of σ at x.[27]

It is easy to see that every local mapping is almost local.

Theorem F.30 (Smoothness Theorem) *Let $f \colon C^r_c(M, E_1) \supseteq P \to C^s_c(N, E_2)$ be a map as described in Definition* F.29*. If $f_K := f|_{P\cap C^r_K(M,E_1)}$ is of class $C^k_{\mathbb{K}}$ for every compact subset $K \subseteq M$ and f is locally almost local, then f is of class $C^k_{\mathbb{K}}$.*

Proof. Given $\sigma \in P$, there exists an open neighbourhood Q of σ in P such that $f|_Q$ is almost local. As σ was arbitrary, the assertion will follow if we can show that $f|_W$ is of class $C^k_{\mathbb{K}}$ for some open neighbourhood W of σ in Q. To this end, it suffices to show that the mapping $g : Q - \sigma \to C^s_c(N, E_2)$, $g(\tau) := f(\sigma + \tau) - f(\sigma)$ is of class $C^k_{\mathbb{K}}$ on some open zero-neighbourhood. As $f|_Q$ is almost local, we find locally finite covers $(U_i)_{i\in I}$ of M and $(V_i)_{i\in I}$ of N, with each U_i and V_i relatively compact and open, such that $f(\tau)|_{V_i}$ only depends on $\tau|_{U_i}$, for all $\tau \in Q$. Then apparently also $g(\tau)|_{V_i} = g(\kappa)|_{V_i}$ for all $\tau, \kappa \in Q - \sigma$ such that $\tau|_{U_i} = \kappa|_{U_i}$, showing that also g is almost local. Furthermore, given a compact subset $K \subseteq M$, the map $g|_{(Q-\sigma)\cap C^r_K(M,E_1)}$ is of class $C^k_{\mathbb{K}}$, since so is the restriction of f to $Q \cap C^r_{K\cup \mathrm{supp}(\sigma)}(M, E_1)$. We abbreviate $R := Q - \sigma$.

Next, we pick a locally finite open cover $(\widetilde{U}_i)_{i\in I}$ of M such that $\overline{U_i} \subseteq \widetilde{U}_i$ holds for the compact closures, for all $i \in I$; such a "thickening" exists by Lemma 8.5. For each $i \in I$,

[27] More precisely, we require $f(\sigma)(x) = f(\tau)(x)$ for all $x \in M$ and $\sigma, \tau \in P$ with the same germ at x.

we pick a mapping $h_i \in C^r(\widetilde{U}_i, \mathbb{K})$, with compact support $K_i := \operatorname{supp}(h_i)$, which is constantly 1 on U_i (see Lemma 8.8 if $\mathbb{K}$ is a local field; the real case is standard).

By Remark F.21, the family $(\rho_i)_{i\in I}$ of restriction maps $\rho_i : C^r_c(M, E_1) \to C^r(\widetilde{U}_i, E_1|_{\widetilde{U}_i})$ is a patchwork for $C^r_c(M, E_1)$. We let $\rho\colon C^r_c(M, E_1) \to \bigoplus_{i\in I} C^r(\widetilde{U}_i, E_1|_{\widetilde{U}_i}) =: S$ be the corresponding embedding taking τ to $\sum_{i\in I} \rho_i(\tau)$. Similarly, the family $(\xi_i)_{i\in I}$ of restriction maps $\xi_i\colon C^s_c(N, E_2) \to C^s(V_i, E_2|_{V_i})$ is a patchwork for $C^s_c(N, E_2)$.

The mapping ρ being a topological embedding, we find an open 0-neighbourhood $H \subseteq S$ such that $\rho^{-1}(H) \subseteq R$. The direct sum being equipped with the box topology, after shrinking H we may assume that $H = \bigoplus_{i\in I} A_i$ for a family $(A_i)_{i\in I}$ of open 0-neighbourhoods $A_i \subseteq C^r(\widetilde{U}_i, E_1|_{\widetilde{U}_i})$. As a consequence of Corollary F.13, the multiplication operator $\mu_{h_i} : C^r(\widetilde{U}_i, E_1|_{\widetilde{U}_i}) \to C^r_{K_i}(\widetilde{U}_i, E_1|_{\widetilde{U}_i})$, $\tau \mapsto h_i \cdot \tau$ is continuous linear. Hence, we find an open zero-neighbourhood $W_i \subseteq A_i$ such that $h_i \cdot W_i \subseteq R$, where we identify $C^r_{K_i}(\widetilde{U}_i, E_1|_{\widetilde{U}_i})$ with $C^r_{K_i}(M, E_1)$ as a topological $\mathbb{K}$-vector space in the natural way, extending sections by 0 (cf. Lemma F.15 (b)). Then $W := \rho^{-1}(\bigoplus_{i\in I} W_i) \subseteq R$ is an open zero-neighbourhood in $C^r_c(M, E_1)$ such that $\rho_i(W) \subseteq W_i$ for each $i \in I$. We define

$$g_i\colon W_i \to C^s(V_i, E_2|_{V_i}), \quad g_i := \xi_i \circ g|_{R\cap C^r_{K_i}(M,E_1)} \circ \mu_{h_i}|^R_{W_i} .$$

Then g_i is of class $C^k_{\mathbb{K}}$, being a composition of $C^k_{\mathbb{K}}$-maps. Note that $\xi_i(g(\tau)) = g(\tau)|_{V_i} = g(h_i \cdot \tau)|_{V_i} = g_i(\tau|_{\widetilde{U}_i})$ for each $\tau \in W$ and $i \in I$. Thus $(g_i)_{i\in I}$ is compatible with $g|_W$ in the sense of Definition 8.20. We have shown that $g|_W$ is a patched mapping which is of class $C^k_{\mathbb{K}}$ on the patches. By Proposition 8.21, $g|_W$ is of class $C^k_{\mathbb{K}}$. □

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List of Symbols

Standard Symbols:

$\mathbb{C}$ the field of complex numbers
$\mathbb{N} = \{1, 2, 3, \ldots\}$
$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$
$\mathbb{R}$ the field of real numbers
$\mathbb{Q}_p$ the field of p-adic numbers
$\mathbb{S}^1 = \{z \in \mathbb{C} \colon |z| = 1\}$
$\mathbb{Z}$ the integers

Operations on Sets and Maps

Y^0 interior of $Y \subseteq X$ in X
$\overline{Y}$ closure of a subset Y of a topological space (unless re-defined)
$f|_Y$ restriction of f to Y
$f|^Y$ corestriction of f to $Y \supseteq \mathrm{im}(f)$

Special Symbols